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DISTANCE BETWEEN BEHAVIORS AND RATIONAL REPRESENTATIONS*

H. L. TRENTELMAN[†] AND S. V. GOTTIMUKKALA[†]

Dedicated to the memory of Jan C. Willems (1939–2013)

Abstract. In this paper we study notions of distance between behaviors of linear differential systems. We introduce four metrics on the space of all controllable behaviors which generalize existing metrics on the space of input-output systems represented by transfer matrices. Three of these are defined in terms of gaps between closed subspaces of the Hilbert space $\mathcal{L}_2(\mathbb{R})$. In particular we generalize the “classical” gap metric. We express these metrics in terms of *rational representations* of behaviors. In order to do so, we establish a precise relation between rational representations of behaviors and multiplication operators on $\mathcal{L}_2(\mathbb{R})$. We introduce a fourth behavioral metric as a generalization of the well-known ν -metric. As in the input-output framework, this definition is given in terms of rational representations. For this metric, however, we establish a representation-free, behavioral characterization as well. We make a comparison between the four metrics and compare the values they take and the topologies they induce. Finally, for all metrics we make a detailed study of necessary and sufficient conditions under which the distance between two behaviors is less than one. For this, both behavioral as well as state space conditions are derived in terms of driving variable representations of the behaviors.

Key words. behaviors, linear differential systems, gap metric, ν -metric, rational representations, multiplication operators

AMS subject classifications. 93B20, 93B28, 93B36, 93C05

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1. Introduction. This paper deals with notions of *distance* between systems. In the context of linear systems with inputs and outputs, several concepts of distance have been studied in the past. Perhaps the most well-known distance concept is that of *gap metric* introduced by Zames and El-Sakkary in [28] and extensively used by Georgiou and Smith in the context of robust stability in [7]. The distance between two systems in the gap metric can be calculated, but the calculation is by no means easy and requires the solution of an \mathcal{H}_∞ optimization problem; see [6]. A distance concept which is equally relevant in the context of robust stability is the so-called ν -gap, introduced by Vinnicombe in [22], [21]. Computation of the ν -gap between two systems is much easier than that of the ordinary gap and basically requires computation of the *winding number* of a certain proper rational function, followed by computation of the \mathcal{L}_∞ -norm of a given proper rational matrix. A third distance concept is that of \mathcal{L}_2 -gap, which is the most easy to compute but which is not at all useful in the context of robust stability, as shown in [21]. More recently an alternative notion of gap for linear input-output systems was introduced by Ball and Sasane in [13], allowing also nonzero initial conditions of the system.

In this paper we will put the above four distance concepts into a more general, behavioral context, extending them to a framework in which the systems are not necessarily identified with their representations (e.g., transfer matrices), but in which, instead, their *behaviors*, i.e., the spaces of all possible trajectories of the systems, form

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the core of the theory. This idea was put forward for the first time in [12]. Indeed, we will introduce four metrics on the set of all (controllable) behaviors with a fixed number of variables that we will call the \mathcal{L}_2 -metric, the Zames (Z) metric, the Sasane–Ball (SB) metric, and the Vinnicombe (V) metric. The first three will be defined in terms of the concept of “gap” between closed subspaces of the Hilbert space $\mathcal{L}_2(\mathbb{R}, \mathbb{C}^q)$ of square integrable functions; the fourth one, the V-metric, will be defined in terms of representations of the behaviors. Of course, no a priori input-output partition of the system variables needs to be given. Our setup will, however, be applicable also to the “classical” input-output framework. We will establish several behavioral, representation-free characterizations of properties of the metrics we have introduced. We will also study the interrelation between the metrics and compare the topologies they induce.

We want to mention that the idea of distance between behaviors was also studied in a more general framework in [3]. The latter paper deals with behaviors as general *subsets* of the set of all functions from time axis to signal space (not necessarily representable by higher order linear differential equations) and introduces a notion of distance between such behaviors.

A key ingredient in our paper will be the notion of *rational representation* of behaviors, recently introduced in [27]. Whereas, originally, behaviors of linear differential systems were defined as kernels and images of polynomial differential operators, in [27] it was explained how they also allow representations as “kernels” and “images” of “rational differential operators” in a mathematically consistent, natural way. In fact, for a given behavior, there is freedom in the choice of the rational matrices used for its representation, and they can, for example, be chosen to be proper, bounded on the imaginary axis, stable, prime, and inner, all at the same time. In this paper we will use these properties of the rational representations to describe the relationship between kernels and images of the rational differential operators on the one hand and kernels and images of the (operator theoretic) multiplication operators associated with the rational matrices on the other.

Using the relation between rational representations of behaviors and multiplication operators, we will on the one hand express the \mathcal{L}_2 -metric, Z-metric, and SB-metric in terms of rational representations, and on the other hand give a representation-free characterization of the V-metric. As a special case, this will provide a representation-free characterization of the classical ν gap in the input-output framework.

For each of the four metrics we will also characterize under which conditions the distance between two behaviors is strictly less than one. For the \mathcal{L}_2 -metric and the V-metric this will turn out to be relatively easy, and we obtain behavioral characterizations for this. However, for the Z-metric and the SB-metric this is more involved, and we will make a detailed study of this problem using driving variable state representations of the behaviors involved. This will also involve the problem of state representation of the kernel of a Toeplitz operator with an invertible symbol.

The outline of this paper is as follows. In section 2 we review behaviors of linear differential systems and introduce the \mathcal{L}_2 -metric, Z-metric, and SB-metric. In section 3 we briefly review rational kernel and image representations of behaviors. We also show that behaviors admit rational image representations in which the rational matrices are proper and stable, right prime, and inner and have no zeros. An analogous result is proved for rational kernel representations. In section 4 we establish in detail the relation between rational image and kernel representations of behaviors on the one hand and the images and kernels of the classical multiplication operators associated with these representations on the other. Using this relation, in section 5 we express

all three behavioral metrics that were introduced in section 2 in terms of rational representations of the behaviors. We also show that our definitions of Z-metric and SB-metric generalize classical gap metrics for input-output systems represented by transfer matrices. In section 6 we introduce the fourth metric, the V-metric. Unlike the other three metrics, the definition of this metric is in terms of representations of the behaviors, involving the notion of winding number. We will in this section derive a new, representation-free, behavioral characterization of this metric. Section 7 deals with a comparison of the four metrics. We will compare both the values they take and the topologies they induce. In section 8, for each of the metrics we find conditions under which the distance between two behaviors is strictly less than the value one. For the \mathcal{L}_2 -metric and the V-metric this issue is readily dealt with, and we obtain behavioral characterizations. For the Z-metric and the SB-metric this is a harder problem, and in sections 9 and 10 we study this problem using driving variable state representations of the behaviors. This also involves the study of Toeplitz operators with an invertible symbol. The paper closes with conclusions in section 11.

1.1. Basic concepts and notation. We now introduce the basic concepts and notation used in this paper. We will denote the ring of polynomials with real coefficients by $\mathbb{R}[\xi]$. The field of real rational functions is denoted by $\mathbb{R}(\xi)$ and the ring of proper real rational functions by $\mathbb{R}(\xi)_P$. As usual, a proper real rational function will be called stable if its poles are in $\mathbb{C}^- := \{\lambda \in \mathbb{C} \mid \operatorname{Re}(\lambda) < 0\}$. It is called antistable if its poles are in $\mathbb{C}^+ := \{\lambda \in \mathbb{C} \mid \operatorname{Re}(\lambda) > 0\}$. $R\mathcal{L}_\infty$ will denote the ring of all proper real rational functions without poles on the imaginary axis, and $R\mathcal{H}_\infty$ denotes the ring of all proper and stable real rational matrices. $R\mathcal{H}_\infty^-$ will denote the ring of proper antistable real rational functions.

For a given ring \mathcal{R} , a matrix G with coefficients in \mathcal{R} is called right prime over \mathcal{R} (left prime over \mathcal{R}) if there exists a matrix G^+ with coefficients in \mathcal{R} such that $G^+G = I$ ($GG^+ = I$). In this paper the condition of primeness will occur with respect to the rings $\mathbb{R}[\xi]$, $\mathbb{R}(\xi)_P$, $R\mathcal{L}_\infty$, $R\mathcal{H}_\infty$, and $R\mathcal{H}_\infty^-$. We will be using matrices with coefficients from the above rings. In order to streamline notation we will suppress the dimensions. For example, the spaces of all real rational matrices with coefficients in $R\mathcal{L}_\infty$ or $R\mathcal{H}_\infty$ will again be denoted by $R\mathcal{L}_\infty$ or $R\mathcal{H}_\infty$.

For a given real rational matrix G we denote $G^*(\xi) := G^\top(-\xi)$. A proper real rational matrix is called inner if $G^*G = I$ and co-inner if $GG^* = I$. Note that if $G \in R\mathcal{L}_\infty$ is inner (co-inner), then it is right prime (left prime) over $R\mathcal{L}_\infty$. The analogous statement is not true for left and right primeness over $R\mathcal{H}_\infty$. G is called unitary if $G^*G = GG^* = I$. We denote the usual infinity norm of $G \in R\mathcal{L}_\infty$ by $\|G\|_\infty$. For a given complex matrix M , $\sigma_{\max}M$ and $\sigma_{\min}M$ denote the largest and smallest singular value, respectively. Note that $\|G\|_\infty = \sup_{\omega \in \mathbb{R}} \sigma_{\max} G(i\omega)$. For a given real rational matrix G , its *zeros* are the roots of the nonzero numerator polynomials in the Smith–McMillan form of G (see [27]).

For a given real rational function g without poles or zeros on the imaginary axis, the *winding number* of g is defined as the net number of counterclockwise encirclements of the origin by the (closed) contour $g(\lambda)$ as λ traverses in counterclockwise direction a standard D -contour enclosing all poles and zeros of g in \mathbb{C}^+ . The winding number of g is denoted by $\operatorname{wno}(g)$, and is equal to the difference $Z - P$, where Z is the number of zeros and P is the number of poles of g in \mathbb{C}^+ .

In this paper, we will only consider real-valued signals. For a given integer q , we denote by $\mathcal{L}_2(\mathbb{R}, \mathbb{R}^q)$ the space of all Lebesgue measurable functions $w : \mathbb{R} \rightarrow \mathbb{R}^q$ such that $\int_{-\infty}^{\infty} \|w(t)\|^2 dt < \infty$. This is a Hilbert space with inner product given by

$\langle w_1, w_2 \rangle = \int_{-\infty}^{\infty} w_1(t)^\top w_2(t) dt$. The usual \mathcal{L}_2 -norm is denoted by $\|\cdot\|_2$. In what follows, in the notation we will mostly suppress the dimension q and simply denote this Hilbert space by $\mathcal{L}_2(\mathbb{R})$. The subset of all signals w such that $w(t) = 0$ for almost all $t < 0$ is a closed subspace of $\mathcal{L}_2(\mathbb{R})$ and is denoted by $\mathcal{L}_2(\mathbb{R}^+)$. Likewise, $\mathcal{L}_2(\mathbb{R}^-)$ will denote the closed subspace consisting of all signals w such that $w(t) = 0$ for almost all $t > 0$. Obviously, $\mathcal{L}_2(\mathbb{R}^-)^\perp = \mathcal{L}_2(\mathbb{R}^+)$. The orthogonal projections of $\mathcal{L}_2(\mathbb{R})$ onto $\mathcal{L}_2(\mathbb{R}^+)$ and $\mathcal{L}_2(\mathbb{R}^-)$ are denoted by Π_+ and Π_- , respectively.

In addition to their time-domain descriptions, signals allow descriptions in the frequency domain. For given integer q , we denote by $\mathcal{L}_2(i\mathbb{R}, \mathbb{C}^q)$ the space of all Lebesgue measurable functions $W : i\mathbb{R} \rightarrow \mathbb{C}^q$ such that $\frac{1}{2\pi} \int_{-\infty}^{\infty} \|W(i\omega)\|^2 d\omega < \infty$. This is again a Hilbert space with inner product given by $\langle W_1, W_2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} W_1(i\omega)^* W_2(i\omega) d\omega$. Again, we suppress the dimension q in the notation and denote this space by $\mathcal{L}_2(i\mathbb{R})$. We will denote the usual Hardy space of all complex valued functions W that are analytic in \mathbb{C}^+ and that satisfy $\sup_{\sigma>0} \frac{1}{2\pi} \int_{-\infty}^{\infty} W_1(\sigma+i\omega)^* W_2(\sigma+i\omega) d\omega < \infty$ by \mathcal{H}_2 . This space can be identified with a closed subspace of $\mathcal{L}_2(i\mathbb{R})$ (see [5]).

The usual Fourier transformation is denoted by \mathcal{F} . The Fourier transform $W(i\omega)$ of a (real-valued) signal $w \in \mathcal{L}_2(\mathbb{R})$ satisfies the property $W(-i\omega) = \overline{W(i\omega)}$, where \overline{v} denotes the componentwise complex conjugate of $v \in \mathbb{C}^q$. Define

$$(1.1) \quad \mathcal{S} := \{W \in \mathcal{L}_2(i\mathbb{R}, \mathbb{C}^q) \mid W(-i\omega) = \overline{W(i\omega)} \ \forall \omega \in \mathbb{R}\}.$$

It is well known that \mathcal{F} is a linear transformation and that it defines a bijection between $\mathcal{L}_2(\mathbb{R})$ and the subspace \mathcal{S} . Furthermore, $\mathcal{F}\mathcal{L}_2(\mathbb{R}^+) = \mathcal{H}_2 \cap \mathcal{S}$ and $\mathcal{F}\mathcal{L}_2(\mathbb{R}^-) = \mathcal{H}_2^\perp \cap \mathcal{S}$. The inverse of \mathcal{F} will be denoted by \mathcal{F}^{-1} .

We will denote by $\mathcal{L}_{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$ the space of all measurable functions w from \mathbb{R} to \mathbb{R}^q that are locally integrable, i.e., for all t_0, t_1 the integral $\int_{t_0}^{t_1} \|w(t)\| dt$ is finite. For systems of linear differential equations $R(\frac{d}{dt})w = 0$, solutions w are understood to be in this space, and the differential equation is understood to be satisfied in the distributional sense. If the dimensions are clear from the context we use the notation \mathcal{L}_{loc} .

2. Distance between behaviors. In the behavioral context, a linear differential system is defined as a triple $\Sigma = (\mathbb{R}, \mathbb{R}^q, \mathfrak{B})$ with \mathbb{R} the *time axis*, \mathbb{R}^q the *signal space*, and $\mathfrak{B} \subset \mathcal{L}_{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$ the *behavior*, which is equal to the space of solutions of a finite number of higher order, linear, constant coefficient differential equations. For any such system there exists a real polynomial matrix R such that \mathfrak{B} is equal to the space of solutions of the system of differential equations $R(\frac{d}{dt})w = 0$. This is then called a *polynomial kernel representation* of the behavior \mathfrak{B} and we write $\mathfrak{B} = \ker R(\frac{d}{dt})$. The set of all linear differential systems with q variables is denoted by \mathcal{L}^q . The subset of all controllable ones is denoted by $\mathcal{L}_{\text{cont}}^q$. We denote by $\mathfrak{m}(\mathfrak{B})$ (the *input cardinality*) the number of inputs of \mathfrak{B} . For an overview of the basic material on behaviors, we refer to [11], [26].

In this section we will introduce three metrics on the space $\mathcal{L}_{\text{cont}}^q$ of behaviors of controllable linear differential systems, inspired by the several notions of “gap” in the context of input-output systems represented by transfer matrices. The general idea is to associate with every controllable behavior \mathfrak{B} a suitable subspace of the Hilbert space $\mathcal{L}_2(\mathbb{R})$ and in this way define a metric on $\mathcal{L}_{\text{cont}}^q$ in terms of the usual metric on the set of closed subspaces of $\mathcal{L}_2(\mathbb{R})$. This can be done in several ways, and each of these choices will lead to a particular metric on $\mathcal{L}_{\text{cont}}^q$. In later sections, we will study these metrics and compare them.

In order to set the scene, we now first review some standard material on the gap between closed subspaces of a Hilbert space (see, e.g., [1] or [23]). For a given Hilbert space \mathcal{H} , the *directed gap* between two closed subspaces \mathcal{V}_1 and \mathcal{V}_2 of \mathcal{H} is defined as

$$\text{gap}(\mathcal{V}_1, \mathcal{V}_2) := \sup_{\substack{v_1 \in \mathcal{V}_1 \\ \|v_1\|=1}} \inf_{v_2 \in \mathcal{V}_2} \|v_1 - v_2\|.$$

The *gap* between \mathcal{V}_1 and \mathcal{V}_2 is then defined as

$$\text{gap}(\mathcal{V}_1, \mathcal{V}_2) := \max(\text{gap}(\mathcal{V}_1, \mathcal{V}_2), \text{gap}(\mathcal{V}_2, \mathcal{V}_1)).$$

The gap between two subspaces always lies between zero and one, i.e., $0 \leq \text{gap}(\mathcal{V}_1, \mathcal{V}_2) \leq 1$ for all $\mathcal{V}_1, \mathcal{V}_2$. It is also well known that the gap between two subspaces can be expressed in terms of the norms of the orthogonal projection operators onto these subspaces. More specific, $\text{gap}(\mathcal{V}_1, \mathcal{V}_2) = \|\Pi_{\mathcal{V}_2^\perp} \Pi_{\mathcal{V}_1}\|$. Here, $\Pi_{\mathcal{V}}$ is the orthogonal projection of \mathcal{H} onto \mathcal{V} . Also, $\text{gap}(\mathcal{V}_1, \mathcal{V}_2) = \|\Pi_{\mathcal{V}_1} - \Pi_{\mathcal{V}_2}\|$. Another relevant fact is that the gap does not change after taking orthogonal complements; in other words, $\text{gap}(\mathcal{V}_1, \mathcal{V}_2) = \text{gap}(\mathcal{V}_1^\perp, \mathcal{V}_2^\perp)$.

In this paper, the relevant Hilbert space will always be $\mathcal{H} = \mathcal{L}_2(\mathbb{R})$. The directed gap and gap between two closed linear subspaces of the Hilbert space $\mathcal{L}_2(\mathbb{R})$ will be denoted by $\text{gap}_{\mathcal{L}_2}$ and $\text{gap}_{\mathcal{L}_2}$, respectively.

We now introduce the following metrics on the space $\mathcal{L}_{\text{cont}}^q$ of controllable linear differential systems.

2.1. \mathcal{L}_2 -metric. The first metric that we consider is the one that is directly induced by the gap on the Hilbert space $\mathcal{L}_2(\mathbb{R})$. We will call it the \mathcal{L}_2 -metric.

DEFINITION 2.1. Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$. The \mathcal{L}_2 -metric, denoted by $d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2)$, is defined as the gap between $\mathfrak{B}_1 \cap \mathcal{L}_2(\mathbb{R})$ and $\mathfrak{B}_2 \cap \mathcal{L}_2(\mathbb{R})$ in $\mathcal{L}_2(\mathbb{R})$:

$$d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) := \text{gap}_{\mathcal{L}_2}(\mathfrak{B}_1 \cap \mathcal{L}_2(\mathbb{R}), \mathfrak{B}_2 \cap \mathcal{L}_2(\mathbb{R})).$$

The \mathcal{L}_2 -metric measures the distance between two behaviors as the gap between their \mathcal{L}_2 -behaviors over the whole real line.

2.2. Zames metric. The second metric that we introduce is obtained by intersecting the behaviors with the subspace $\mathcal{L}_2(\mathbb{R}^+)$ of all signals that are zero in the past. We will call it the *Zames metric* because in the input-output transfer matrix context it will turn out to coincide with the classical gap metric.

DEFINITION 2.2. Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$. The *Zames metric*, denoted by $d_Z(\mathfrak{B}_1, \mathfrak{B}_2)$, is defined as the gap between $\mathfrak{B}_1 \cap \mathcal{L}_2(\mathbb{R}^+)$ and $\mathfrak{B}_2 \cap \mathcal{L}_2(\mathbb{R}^+)$ in the Hilbert space $\mathcal{L}_2(\mathbb{R})$:

$$d_Z(\mathfrak{B}_1, \mathfrak{B}_2) := \text{gap}_{\mathcal{L}_2}(\mathfrak{B}_1 \cap \mathcal{L}_2(\mathbb{R}^+), \mathfrak{B}_2 \cap \mathcal{L}_2(\mathbb{R}^+)).$$

In what follows we will often use the shorthand terminology *Z-metric*.

2.3. Sasane–Ball metric. A third metric that we will consider is obtained by projecting the \mathcal{L}_2 -behaviors onto the future and subsequently taking their gap in the Hilbert space $\mathcal{L}_2(\mathbb{R})$. It will be called the *Sasane–Ball metric* since it will turn out to coincide with the behavioral distance introduced in the input-output framework in [13]. Recall that Π_+ is the orthogonal projection of $\mathcal{L}_2(\mathbb{R})$ onto $\mathcal{L}_2(\mathbb{R}^+)$.

DEFINITION 2.3. Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$. The Sasane–Ball metric, denoted by $d_{SB}(\mathfrak{B}_1, \mathfrak{B}_2)$, is defined as the gap between $\Pi_+(\mathfrak{B}_1 \cap \mathcal{L}_2(\mathbb{R}))$ and $\Pi_+(\mathfrak{B}_2 \cap \mathcal{L}_2(\mathbb{R}))$ in $\mathcal{L}_2(\mathbb{R})$:

$$d_{SB}(\mathfrak{B}_1, \mathfrak{B}_2) := \text{gap}_{\mathcal{L}_2}(\Pi_+(\mathfrak{B}_1 \cap \mathcal{L}_2(\mathbb{R})), \Pi_+(\mathfrak{B}_2 \cap \mathcal{L}_2(\mathbb{R}))).$$

The Sasane–Ball metric measures the distance over the future time axis with arbitrary past. The Hilbert space is again taken as $\mathcal{L}_2(\mathbb{R})$. We will often use the shorthand terminology *SB-metric*.

In what follows we will make a detailed study of the above three metrics and express their properties in terms of rational representations of the behaviors and their associated multiplication operators. Later, we will also introduce a fourth metric, the *Vinnicombe metric*. As in the input-output case, the definition of the latter can only be given in terms of representations, since it does not seem to allow a natural interpretation in terms of gap in Hilbert space.

3. Rational representations of behaviors. In addition to polynomial representations, behaviors admit *rational representations* (see [27]). In particular, for a real rational matrix G a meaning can be given to the equation $G(\frac{d}{dt})w = 0$ and to the expression $\ker G(\frac{d}{dt})$. For this we need the concept of left coprime factorization of a rational matrix G over $\mathbb{R}[\xi]$. A factorization of such G as $G = P^{-1}Q$ with P and Q real polynomial matrices is called a left coprime factorization if $\begin{pmatrix} P & Q \end{pmatrix}$ is left prime over $\mathbb{R}[\xi]$ and $\det(P) \neq 0$. Following [27], if $G = P^{-1}Q$ is a left coprime factorization over $\mathbb{R}[\xi]$, then we define $G(\frac{d}{dt})w = 0$ if $Q(\frac{d}{dt})w = 0$ and

$$(3.1) \quad \ker G\left(\frac{d}{dt}\right) := \ker Q\left(\frac{d}{dt}\right).$$

In this way every linear differential system also admits representations as the “kernel of a rational matrix.” If G is a rational matrix, we call a representation of \mathfrak{B} as $\mathfrak{B} = \{w \in \mathcal{L}_{\text{loc}}(\mathbb{R}, \mathbb{R}^q) \mid G(\frac{d}{dt})w = 0\}$ a *rational kernel representation* of \mathfrak{B} and often write $\mathfrak{B} = \ker G(\frac{d}{dt})$. For a more detailed exposition on rational representations of behaviors see [27], [15].

Therein, it can also be found that a linear differential system is controllable if and only if its behavior \mathfrak{B} admits a representation

$$(3.2) \quad \mathfrak{B} = \left\{ w \in \mathcal{L}_{\text{loc}}(\mathbb{R}, \mathbb{R}^q) \mid \exists v \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^m) \text{ such that } w = G\left(\frac{d}{dt}\right)v \right\}$$

for some integer m and some real rational matrix G with m columns. The equation $w = G(\frac{d}{dt})v$ should be interpreted as $(I - G(\frac{d}{dt}))\begin{pmatrix} w \\ v \end{pmatrix} = 0$, whose meaning was defined above. The representation (3.2) is called a *rational image representation*, and we often write $\mathfrak{B} = \text{im } G(\frac{d}{dt})$. The minimal m required can be shown to be equal to $\mathfrak{m}(\mathfrak{B})$, the input cardinality of \mathfrak{B} , and is achieved if and only if G has full column rank.

If G is a real rational matrix, then for the behavior $\mathfrak{B} = \text{im } G(\frac{d}{dt})$ we can obtain a *polynomial image representation* as follows, using right coprime factorization over $\mathbb{R}[\xi]$. A factorization $G = MN^{-1}$ with M and N real polynomial matrices is called a right coprime factorization over $\mathbb{R}[\xi]$ if $\begin{pmatrix} M \\ N \end{pmatrix}$ is right prime over $\mathbb{R}[\xi]$ and $\det(N) \neq 0$.

LEMMA 3.1. Let G be a real rational matrix and let $G = MN^{-1}$ be a right coprime factorization over $\mathbb{R}[\xi]$. Then $\text{im } G(\frac{d}{dt}) = \text{im } M(\frac{d}{dt})$.

Proof. Let $G = P^{-1}Q$ be a left coprime factorization over $\mathbb{R}[\xi]$. Then we obviously have $(P \quad -Q) \begin{pmatrix} M \\ N \end{pmatrix} = 0$. Using this, it can be shown that

$$\ker \begin{pmatrix} P \left(\frac{d}{dt} \right) & -Q \left(\frac{d}{dt} \right) \end{pmatrix} = \text{im} \begin{pmatrix} M \left(\frac{d}{dt} \right) \\ N \left(\frac{d}{dt} \right) \end{pmatrix}$$

(see [19, Proposition 3.2]). Since $(I - G) = P^{-1}(P - Q)$ is a left coprime factorization, we have $w = G(\frac{d}{dt})v \Leftrightarrow (I - G(\frac{d}{dt}))\text{col}(w, v) = 0 \Leftrightarrow (P(\frac{d}{dt}) - Q(\frac{d}{dt}))\text{col}(w, v) = 0$. Thus we obtain $w \in \text{im } G(\frac{d}{dt}) \Leftrightarrow \exists v$ such that $w = G(\frac{d}{dt})v$. In turn this is equivalent with: $\exists v$ such that $(P(\frac{d}{dt}) - Q(\frac{d}{dt}))\text{col}(w, v) = 0 \Leftrightarrow \exists v, \ell$ such that $\begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} M(\frac{d}{dt}) \\ N(\frac{d}{dt}) \end{pmatrix} \ell \Leftrightarrow \exists \ell$ such that $w = M(\frac{d}{dt})\ell \Leftrightarrow w \in \text{im } M(\frac{d}{dt})$. \square

A given behavior allows rational image and kernel representations in which the rational matrices satisfy certain desired properties. In particular, they can be chosen to be proper, stable, prime and (co-)inner at the same time. The precise statement is as follows.

THEOREM 3.2. *Let $\mathfrak{B} \in \mathcal{L}_{\text{cont}}^q$. There exists a real rational matrix G such that $\mathfrak{B} = \text{im } G(\frac{d}{dt})$, where G satisfies the following four properties:*

1. $G \in R\mathcal{H}_{\infty}$,
2. G is right prime over $R\mathcal{H}_{\infty}$,
3. G is inner,
4. G has no zeros.

If $\mathfrak{B} \in \mathcal{L}^q$, then there exists a real rational matrix \tilde{G} such that $\mathfrak{B} = \ker \tilde{G}(\frac{d}{dt})$, where \tilde{G} satisfies the following three properties:

1. $\tilde{G} \in R\mathcal{H}_{\infty}$,
2. \tilde{G} is left prime over $R\mathcal{H}_{\infty}$,
3. \tilde{G} is co-inner.

Proof. We first prove the existence of G satisfying properties 1, 2, 3, and 4 such that $\mathfrak{B} = \text{im } G(\frac{d}{dt})$. Since $\mathfrak{B} \in \mathcal{L}_{\text{cont}}^q$, by [27, Theorem 9] there exists $G_1 \in R\mathcal{H}_{\infty}$, right prime over $R\mathcal{H}_{\infty}$ and having no zeros, such that $\mathfrak{B} = \text{im } G_1(\frac{d}{dt})$. We now adapt G_1 in such a way that also property 3 is satisfied.

Define $Z := G_1^* G_1$. By right primeness of G_1 it is easily verified that Z is biproper. Further we have $Z^* = Z$ and Z has no poles and zeros on the imaginary axis. Let $G_1 = MN^{-1}$ be a right coprime factorization over $\mathbb{R}[\xi]$. Now let L be a square polynomial matrix such that $M^*M = L^*L$, and L is Hurwitz. Indeed such L exists and is obtained by polynomial spectral factorization of M^*M . Define $W := LN^{-1}$. Since Z is biproper, we have

$$\deg \det(N^*N) = \deg \det(M^*M) = \deg \det(L^*L).$$

Therefore W is biproper and since N and L are both Hurwitz, we have $W, W^{-1} \in R\mathcal{H}_{\infty}$. Define $G := G_1 W^{-1}$. Clearly $G \in R\mathcal{H}_{\infty}$ and $G^*G = I$. Further, since G_1 is right prime over $R\mathcal{H}_{\infty}$, there exists $G_1^+ \in R\mathcal{H}_{\infty}$ such that $G_1^+ G_1 = I$. Define $G^+ := W G_1^+$. Clearly $G^+ \in R\mathcal{H}_{\infty}$ and $G^+ G = I$, so G is right prime over $R\mathcal{H}_{\infty}$. Clearly $G = M L^{-1}$ is a right coprime factorization over $\mathbb{R}[\xi]$. Therefore $\text{im } G(\frac{d}{dt}) = \text{im } M(\frac{d}{dt}) = \text{im } G_1(\frac{d}{dt}) = \mathfrak{B}$.

If $\mathfrak{B} \in \mathcal{L}^q$, from Theorem 5 in [27], it admits a rational kernel representation $\mathfrak{B} = \ker \tilde{G}_1(\frac{d}{dt})$ such that \tilde{G}_1 is right prime over $R\mathcal{H}_{\infty}$. Using polynomial spectral

factorization of $\tilde{G}_1 \tilde{G}_1^*$ a rational matrix \tilde{G} can then be obtained such that $\mathfrak{B} = \ker G(\frac{d}{dt})$ and the conditions are satisfied. \square

Obviously, the above theorem also holds with $R\mathcal{H}_\infty$ replaced by $R\mathcal{H}_\infty^-$, the space of proper and antistable real rational matrices. In this paper we will often use rational image and kernel representations of \mathfrak{B} that satisfy some, or all, of the properties stated in Theorem 3.2. In general, if $\mathfrak{B} = \text{im } G(\frac{d}{dt}) = \ker \tilde{G}(\frac{d}{dt})$, then obviously $\tilde{G}G = 0$. If, in addition, G is inner and \tilde{G} is co-inner, then it is immediate that the rational matrix $(G \ \tilde{G}^*)$ is unitary and therefore also $GG^* + \tilde{G}^*\tilde{G} = I$.

To conclude this section, we review the notion of dual behavior; see [18, section 10] and [20]. For a given behavior $\mathfrak{B} \in \mathcal{L}_{\text{cont}}^q$ we define its *dual* behavior \mathfrak{B}^* by

$$\mathfrak{B}^* := \left\{ w \in \mathcal{L}_{\text{loc}}(\mathbb{R}, \mathbb{R}^q) \mid \int_{-\infty}^{\infty} w(t)^\top w'(t) dt = 0 \ \forall w' \in \mathfrak{B} \text{ with compact support} \right\}.$$

It can be shown that $\mathfrak{B}^* \in \mathcal{L}_{\text{cont}}^q$ and that $\mathfrak{m}(\mathfrak{B}^*) = q - \mathfrak{m}(\mathfrak{B})$. Moreover, in the polynomial context $\mathfrak{B} = \text{im } M(\frac{d}{dt})$ if and only if $\mathfrak{B}^* = \ker M^*(\frac{d}{dt})$. Using Lemma 3.1, this carries over to rational representations: for real rational G we have $\mathfrak{B} = \text{im } G(\frac{d}{dt})$ if and only if $\mathfrak{B}^* = \ker G^*(\frac{d}{dt})$.

4. Rational representations and multiplication operators. In this section we will study the relation between rational representations of behaviors and classical multiplication operators on $\mathcal{L}_2(\mathbb{R})$. In particular we will clarify the connection between rational kernel and image representations and the kernels and images of the associated multiplication operators.

With any real rational matrix $G \in R\mathcal{L}_\infty$ we can associate a unique linear operator $G : \mathcal{L}_2(i\mathbb{R}) \rightarrow \mathcal{L}_2(i\mathbb{R})$ whose action is defined by the multiplication $W \mapsto GW$. If $G \in R\mathcal{H}_\infty$, then the subspace \mathcal{H}_2 is invariant under the multiplication operator, i.e., $G\mathcal{H}_2 \subset \mathcal{H}_2$. In this paper we will focus on system descriptions in the time domain. Let \mathcal{F} denote the Fourier transformation. Then, with any $G \in R\mathcal{L}_\infty$ we associate a time-domain “multiplication operator” in the usual way as follows.

DEFINITION 4.1. Let $G \in R\mathcal{L}_\infty$. The operator $M_G : \mathcal{L}_2(\mathbb{R}) \rightarrow \mathcal{L}_2(\mathbb{R})$ is defined by $M_G := \mathcal{F}^{-1}G\mathcal{F}$.

We will call M_G the *multiplication operator with symbol G* . Of course, M_G can be interpreted as a convolution operator, but we will not use this fact here. Obviously, if $G_1, G_2 \in R\mathcal{L}_\infty$, then $M_{G_1 G_2} = M_{G_1} M_{G_2}$. It is a well-known fact that the operator M_G is an isometry, i.e., $\|M_G w\|_2 = \|w\|_2$ for all $w \in \mathcal{L}_2(\mathbb{R})$ if and only if G is inner, i.e., $G^*G = I$. For any $G \in R\mathcal{L}_\infty$, the operator norm $\|M_G\|$ is equal to the \mathcal{L}_∞ -norm $\|G\|_\infty$. Also, if $G_1, G_2, G_3 \in R\mathcal{L}_\infty$ and G_1 is inner and G_3 is co-inner, then $\|M_{G_1} M_{G_2} M_{G_3}\| = \|M_{G_2}\|$. The restriction of M_G to $\mathcal{L}_2(\mathbb{R}^+)$ is denoted by $M_G|_{\mathcal{L}_2(\mathbb{R}^+)}$. The composition $\Pi_+ M_G|_{\mathcal{L}_2(\mathbb{R}^+)} : \mathcal{L}_2(\mathbb{R}^+) \rightarrow \mathcal{L}_2(\mathbb{R}^+)$ is called the Toeplitz operator with symbol G . It will be denoted by T_G . If $G \in R\mathcal{H}_\infty$, then $\mathcal{L}_2(\mathbb{R}^+)$ is invariant under M_G . In this case, M_G is called *causal*. Also, then $M_G|_{\mathcal{L}_2(\mathbb{R}^+)} = T_G$.

We will now study the connection between rational representations of behaviors and multiplication operators. In particular, with any $p \times q$ real rational matrix $G \in R\mathcal{L}_\infty$ we can associate the linear differential behaviors $\ker G(\frac{d}{dt}) \subset \mathcal{L}_{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$ and $\text{im } G(\frac{d}{dt}) \subset \mathcal{L}_{\text{loc}}(\mathbb{R}, \mathbb{R}^p)$. On the other hand G defines a multiplication operator M_G with $\ker M_G \subset \mathcal{L}_2(\mathbb{R})$ and $\text{im } M_G \subset \mathcal{L}_2(\mathbb{R})$. We will now study the relation between these different kernels and images. We first prove the following useful lemma.

LEMMA 4.2. Let $G, \tilde{G} \in R\mathcal{L}_\infty$ be such that $\text{im } G(\frac{d}{dt}) = \ker \tilde{G}(\frac{d}{dt})$. If G is right-prime and \tilde{G} is left-prime (over $R\mathcal{L}_\infty$), then $\text{im } M_G = \ker M_{\tilde{G}}$. If $G, \tilde{G} \in R\mathcal{H}_\infty$ and G is right-prime and \tilde{G} is left-prime (over $R\mathcal{H}_\infty$), then $\text{im } T_G = \ker T_{\tilde{G}}$.

Proof. Since $G\tilde{G} = 0$ we have $M_{\tilde{G}}M_G = 0$, whence $\text{im } M_G \subset \ker M_{\tilde{G}}$. Conversely, let $G^+, \tilde{G}^+ \in R\mathcal{L}_\infty$ such that $G^+G = I$ and $\tilde{G}\tilde{G}^+ = I$. Then clearly

$$\begin{pmatrix} G^+ \\ \tilde{G} \end{pmatrix} (G \quad \tilde{G}^+) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

whence $GG^+ + \tilde{G}^+\tilde{G} = I$. Then $M_{\tilde{G}}w = 0$ implies $w = M_G M_{G^+}w$. Since $M_{G^+}w \in \mathcal{L}_2(\mathbb{R})$, the result follows. The second statement follows in a similar manner. \square

The above lemma is instrumental in proving the following basic relation between rational representations and multiplication operators.

THEOREM 4.3. *Let $G \in R\mathcal{L}_\infty$. Then the following hold:*

1. $\ker G(\frac{d}{dt}) \cap \mathcal{L}_2(\mathbb{R}) = \ker M_G$.
2. *If G is right prime (over $R\mathcal{L}_\infty$), then $\text{im } G(\frac{d}{dt}) \cap \mathcal{L}_2(\mathbb{R}) = \text{im } M_G$.*
3. *If $G \in R\mathcal{H}_\infty$, then $\ker G(\frac{d}{dt}) \cap \mathcal{L}_2(\mathbb{R}^+) = \ker T_G$.*
4. *If $G \in R\mathcal{H}_\infty$ is right prime (over $R\mathcal{H}_\infty$), then $\text{im } G(\frac{d}{dt}) \cap \mathcal{L}_2(\mathbb{R}^+) = \text{im } T_G$.*

Proof. 1. Let $G = P^{-1}Q$ be a right coprime factorization over $\mathbb{R}[\xi]$. Then $w \in \ker G(\frac{d}{dt}) \cap \mathcal{L}_2(\mathbb{R})$ if and only if $Q(\frac{d}{dt})w = 0$ and $w \in \mathcal{L}_2(\mathbb{R})$. This holds if and only if $Q(i\omega)W(i\omega) = 0$ and $W \in \mathcal{S}$, where $W = \mathcal{F}w$ and \mathcal{S} is the subspace of $\mathcal{L}_2(i\mathbb{R})$ given by (1.1). Since P has no roots on the imaginary axis, the latter is equivalent with $P^{-1}(i\omega)Q(i\omega)W(i\omega) = 0$ and $W \in \mathcal{S}$, equivalently, $w \in \mathcal{L}_2(\mathbb{R})$ and $M_Gw = 0$.

2. Let $\tilde{G} \in R\mathcal{L}_\infty$ be left prime and such that $\text{im } G(\frac{d}{dt}) = \ker \tilde{G}(\frac{d}{dt})$. Then, by Lemma 4.2, $\text{im } M_G = \ker M_{\tilde{G}}$, and the result follows from statement 1.

Finally, proofs of 3 and 4 can be given in a similar manner, using a left prime $\tilde{G} \in R\mathcal{H}_\infty$ and with $\mathcal{L}_2(\mathbb{R})$ replaced by $\mathcal{L}_2(\mathbb{R}^+)$ and \mathcal{S} replaced by $\mathcal{H}_2 \cap \mathcal{S}$. \square

In general, for a given behavior \mathfrak{B} , its intersection with $\mathcal{L}_2(\mathbb{R})$ is called an \mathcal{L}_2 -behavior. \mathcal{L}_2 -behaviors have been studied before; see, e.g., [24] or, more recently, [9].

Suitable rational image and kernel representations of a given controllable behavior immediately yield explicit expressions for the orthogonal projection of $\mathcal{L}_2(\mathbb{R})$ onto the associated \mathcal{L}_2 -behavior and its orthogonal complement.

LEMMA 4.4. *Let $\mathfrak{B} \in \mathcal{L}_{\text{cont}}^q$ and let $\mathfrak{B} = \text{im } G(\frac{d}{dt}) = \ker \tilde{G}(\frac{d}{dt})$ with $G, \tilde{G} \in R\mathcal{L}_\infty$ inner and co-inner, respectively. Then the orthogonal projection of $\mathcal{L}_2(\mathbb{R})$ onto $\mathfrak{B} \cap \mathcal{L}_2(\mathbb{R})$ is given by the multiplication operator M_{GG^*} . The orthogonal projection of $\mathcal{L}_2(\mathbb{R})$ onto $(\mathfrak{B} \cap \mathcal{L}_2(\mathbb{R}))^\perp$ is given by the multiplication operator $M_{\tilde{G}^*\tilde{G}}$.*

Proof. In order to prove the first statement note that M_{GG^*} is a projector, $(M_{GG^*})^2 = M_{GG^*}GG^* = M_{GG^*}$; it is self-adjoint, $(M_{GG^*})^* = M_{(GG^*)^*} = M_{GG^*}$; and, by Theorem 4.3, its image $\text{im } M_{GG^*}$ is equal to $\text{im } M_G = \mathfrak{B} \cap \mathcal{L}_2(\mathbb{R})$. The second statement follows from the fact that $M_{\tilde{G}^*\tilde{G}} = I - M_{GG^*}$. \square

A related issue arises if one wants to put the notion of dual behavior in the Hilbert space context and, in particular, relate duality and orthogonality. The following result holds.

LEMMA 4.5. *Let $\mathfrak{B} \in \mathcal{L}_{\text{cont}}^q$. Then $(\mathfrak{B} \cap \mathcal{L}_2(\mathbb{R}))^\perp = \mathfrak{B}^* \cap \mathcal{L}_2(\mathbb{R})$.*

Proof. (\subset) If $w \in \mathcal{L}_2(\mathbb{R})$ satisfies $\int_{-\infty}^{\infty} w^\top(t)w'(t)dt = 0$ for all $w' \in \mathfrak{B} \cap \mathcal{L}_2(\mathbb{R})$, then it does also for all $w' \in \mathfrak{B}$ of compact support. Hence $w \in \mathfrak{B}^* \cap \mathcal{L}_2(\mathbb{R})$.

(\supset) If $w \in \mathfrak{B}^* \cap \mathcal{L}_2(\mathbb{R})$, then $\int_{-\infty}^{\infty} w^\top(t)w'(t)dt = 0$ for all $w' \in \mathfrak{B}$ of compact support. By a density argument (using controllability of \mathfrak{B}) the integral can then be shown to be 0 for all $w \in \mathfrak{B} \cap \mathcal{L}_2(\mathbb{R})$. \square

5. Distance between behaviors and rational representations. Using the relation between rational representations and multiplication operators established in

section 4, in the present section we will for each of the three metrics introduced in section 2 study how to compute their values in terms of rational representations of the behaviors. We will also show that these behavioral metrics are in fact generalizations of classical gaps studied previously in the input-output transfer matrix context.

5.1. \mathcal{L}_2 -metric. Obviously, by Theorem 4.3, if for $i = 1, 2$, $G_i \in R\mathcal{L}_\infty$ is right prime and $\mathfrak{B}_i = \text{im } G_i(\frac{d}{dt})$, and if $\tilde{G}_i \in R\mathcal{L}_\infty$ is such that $\mathfrak{B}_i = \ker \tilde{G}_i(\frac{d}{dt})$, then

$$d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) = \text{gap}_{\mathcal{L}_2}(\text{im } M_{G_1}, \text{im } M_{G_2}) = \text{gap}_{\mathcal{L}_2}(\ker M_{\tilde{G}_1}, \ker M_{\tilde{G}_2}).$$

The following result is well known in the context of input-output systems; see [21], [22]. Here, we state it in the context of rational representations of behaviors, and for completeness we include a proof.

THEOREM 5.1. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$, $\mathfrak{m}(\mathfrak{B}_1) = \mathfrak{m}(\mathfrak{B}_2)$. Let $G_1, G_2 \in R\mathcal{L}_\infty$ such that $\mathfrak{B}_1 = \text{im } G_1(\frac{d}{dt})$ and $\mathfrak{B}_2 = \text{im } G_2(\frac{d}{dt})$ with G_1, G_2 inner. Also, let $\tilde{G}_1, \tilde{G}_2 \in R\mathcal{L}_\infty$ such that $\mathfrak{B}_1 = \ker \tilde{G}_1(\frac{d}{dt})$ and $\mathfrak{B}_2 = \ker \tilde{G}_2(\frac{d}{dt})$ with \tilde{G}_1, \tilde{G}_2 co-inner. Then*

$$d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) = \|\tilde{G}_2 G_1\|_\infty = \|\tilde{G}_1 G_2\|_\infty.$$

Proof. According to Lemma 4.4 we have $\text{gap}_{\mathcal{L}_2}(\mathfrak{B}_1 \cap \mathcal{L}_2(\mathbb{R}), \mathfrak{B}_2 \cap \mathcal{L}_2(\mathbb{R})) = \|M_{\tilde{G}_2^* \tilde{G}_2} M_{G_1 G_1^*}\| = \|M_{G_2^*} M_{\tilde{G}_2 G_1} M_{G_1^*}\| = \|M_{\tilde{G}_2 G_1}\| = \|\tilde{G}_2 G_1\|_\infty$. Thus, $d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) = \max\{\|\tilde{G}_2 G_1\|_\infty, \|\tilde{G}_1 G_2\|_\infty\}$. We prove that $\|\tilde{G}_2 G_1\|_\infty = \|\tilde{G}_1 G_2\|_\infty$. Indeed, using the fact that $G_2 G_2^* + \tilde{G}_2^* \tilde{G}_2 = I$, and pre- and postmultiplying this expression by G_1^* and G_1 , respectively, we see that $(\tilde{G}_2 G_1)^* \tilde{G}_2 G_1 + (G_2^* G_1)^* G_2^* G_1 = I$. This yields $\sigma_{\max}^2(\tilde{G}_2 G_1)(i\omega) = 1 - \sigma_{\min}^2(G_2^* G_1)(i\omega)$ for all $\omega \in \mathbb{R}$. Since the singular values of $(G_2^* G_1)(i\omega)$ and $(G_1^* G_2)(i\omega)$ coincide, this implies $\sigma_{\max}(\tilde{G}_2 G_1)(i\omega) = \sigma_{\max}(\tilde{G}_1 G_2)(i\omega)$ for all ω , whence $\|\tilde{G}_2 G_1\|_\infty = \|\tilde{G}_1 G_2\|_\infty$. \square

Since the gap does not change by taking orthogonal complements in Hilbert space, by applying Lemma 4.5 we immediately obtain that the \mathcal{L}_2 -metric is invariant under dualization of behaviors.

LEMMA 5.2. $d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) = d_{\mathcal{L}_2}(\mathfrak{B}_1^*, \mathfrak{B}_2^*)$.

5.2. Zames metric. We will first show that Definition 2.2 generalizes the classical definition of gap metric in the input-output framework. Indeed, suppose we have two systems, with identical numbers of inputs and outputs, given by their transfer matrices G_1 and G_2 . In [7] the gap $\delta(G_1, G_2)$ is defined as follows. Let $G_i = M_i N_i^{-1}$ be normalized right coprime factorizations with $N_i, M_i \in R\mathcal{H}_\infty$. Then, following [7], the gap between G_1 and G_2 is defined as the \mathcal{L}_2 -gap between the images of the corresponding Toeplitz operators (the “graphs”):

$$\delta(G_1, G_2) = \text{gap}_{\mathcal{L}_2} \left(\text{im } T_{\begin{pmatrix} N_1 \\ M_1 \end{pmatrix}}, \text{im } T_{\begin{pmatrix} N_2 \\ M_2 \end{pmatrix}} \right).$$

This can be interpreted in the behavioral setup as follows. The system with transfer matrix G_i has in fact (input-output) behavior \mathfrak{B}_i given by the rational image representation

$$\begin{pmatrix} u_i \\ y_i \end{pmatrix} = \begin{pmatrix} I \\ G_i(\frac{d}{dt}) \end{pmatrix} v_i.$$

Moreover, by [15, Theorem 7.4], an alternative rational image representation of \mathfrak{B}_i is given by

$$\begin{pmatrix} u_i \\ y_i \end{pmatrix} = \begin{pmatrix} N_i(\frac{d}{dt}) \\ M_i(\frac{d}{dt}) \end{pmatrix} v'_i.$$

By Theorem 4.3 we therefore obtain

$$\delta(G_1, G_2) = \text{gap}_{\mathcal{L}_2}(\mathfrak{B}_1 \cap \mathcal{L}_2(\mathbb{R}^+), \mathfrak{B}_2 \cap \mathcal{L}_2(\mathbb{R}^+)),$$

which indeed equals $d_Z(\mathfrak{B}_1, \mathfrak{B}_2)$ as defined by Definition 2.2. This shows our claim.

In terms of rational representations, the metric defined in Definition 2.2 can be computed in terms of solutions of two \mathcal{H}_∞ optimization problems. The following proposition is a generalization of a well-known result by Georgiou (see [6]) on the computation of gap metric in the input-output framework using normalized coprime factorizations of transfer matrices. We formulate the result here in a general framework using rational representations of behaviors. A proof can be obtained by simply adapting the proof given in [17] in the input-output framework.

PROPOSITION 5.3. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$, $\mathfrak{m}(\mathfrak{B}_1) = \mathfrak{m}(\mathfrak{B}_2)$. Let $G_1, G_2 \in R\mathcal{H}_\infty$ be such that $\mathfrak{B}_1 = \text{im } G_1(\frac{d}{dt})$ and $\mathfrak{B}_2 = \text{im } G_2(\frac{d}{dt})$ with G_1 and G_2 inner and right prime over $R\mathcal{H}_\infty$. Then we have*

$$(5.1) \quad \text{gap}_{\mathcal{L}_2}(\mathfrak{B}_1 \cap \mathcal{L}_2(\mathbb{R}^+), \mathfrak{B}_2 \cap \mathcal{L}_2(\mathbb{R}^+)) = \inf_{Q \in R\mathcal{H}_\infty} \|G_1 - G_2 Q\|_\infty$$

and hence

$$(5.2) \quad d_Z(\mathfrak{B}_1, \mathfrak{B}_2) = \max \left\{ \inf_{Q \in R\mathcal{H}_\infty} \|G_1 - G_2 Q\|_\infty, \inf_{Q \in R\mathcal{H}_\infty} \|G_2 - G_1 Q\|_\infty \right\}.$$

We conclude this subsection with the following result that was obtained in an input-output framework in [21] (see also [17, Theorem 4.7]). The result expresses computation of the distance in the Z-metric as a single optimization problem. The proof from [21] immediately carries over to our framework and will be omitted.

PROPOSITION 5.4. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$, $\mathfrak{m}(\mathfrak{B}_1) = \mathfrak{m}(\mathfrak{B}_2)$. Let $G_1, G_2 \in R\mathcal{H}_\infty$ be such that $\mathfrak{B}_1 = \text{im } G_1(\frac{d}{dt})$ and $\mathfrak{B}_2 = \text{im } G_2(\frac{d}{dt})$ with G_1 and G_2 inner and right prime over $R\mathcal{H}_\infty$. Then*

$$d_Z(\mathfrak{B}_1, \mathfrak{B}_2) = \inf_{Q, Q^{-1} \in R\mathcal{H}_\infty} \|G_1 - G_2 Q\|_\infty.$$

Remark 5.5. As mentioned in the introduction, in [3] a notion of gap between behaviors was introduced in a more general context, with behaviors as arbitrary subsets of the set of all functions from time axis to signal space. This notion of distance was inspired by the gap metric for nonlinear input-output systems introduced in [8]. It can be shown that for the special case of controllable linear differential systems (as is being considered in the present paper) the behavioral gap in [3] specializes to our Zames metric.

5.3. Sasane–Ball metric. In this subsection we show that our definition, Definition 2.3, generalizes the gap as defined by Sasane in [12] and Ball and Sasane in [13]. In [13], for a given minimal input-state-output system $\dot{x} = Ax + Bu$, $y = Cx + Du$ with state space \mathbb{R}^n and stable $p \times m$ transfer matrix G , the “extended graph” is defined as the subspace

$$(5.3) \quad \mathcal{G}(G) := \begin{pmatrix} 0 \\ Ce^{At} \mathbb{1}(t) \end{pmatrix} \mathbb{R}^n + \begin{pmatrix} I \\ T_G \end{pmatrix} \mathcal{L}_2(\mathbb{R}^+, \mathbb{R}^m)$$

of the Hilbert space $\mathcal{L}_2(\mathbb{R}^+, \mathbb{R}^{m+p})$. Here, $\mathbb{1}(t)$ denotes the indicator function of \mathbb{R}^+ and T_G is the Toeplitz operator with symbol G . For stable G the ordinary graph in the gap context is given by

$$\begin{pmatrix} I \\ T_G \end{pmatrix} \mathcal{L}_2(\mathbb{R}^+, \mathbb{R}^m),$$

so the difference lies in the first term in (5.3), which takes into account arbitrary initial conditions on the system. In [13] the following metric is then defined on the space of stable $p \times m$ transfer matrices:

$$\delta'(G_1, G_2) = \text{gap}_{\mathcal{L}_2}(\mathcal{G}(G_1), \mathcal{G}(G_2)).$$

We will now show that for any given transfer matrix G the extended graph is in fact equal to the image of the intersection of the input-output behavior with $\mathcal{L}_2(\mathbb{R})$ under the orthogonal projection onto $\mathcal{L}_2(\mathbb{R}^+)$:

$$(5.4) \quad \mathcal{G}(G) = \Pi_+ \left(\text{im} \begin{pmatrix} I \\ G(\frac{d}{dt}) \end{pmatrix} \cap \mathcal{L}_2(\mathbb{R}) \right).$$

Indeed, from Theorem 4.3 the right-hand side of (5.4) equals

$$\begin{aligned} & \Pi_+ \begin{pmatrix} I \\ M_G \end{pmatrix} \mathcal{L}_2(\mathbb{R}) \\ &= \Pi_+ \begin{pmatrix} I \\ M_G \end{pmatrix} \mathcal{L}_2(\mathbb{R}^-) + \Pi_+ \begin{pmatrix} I \\ M_G \end{pmatrix} \mathcal{L}_2(\mathbb{R}^+) \\ &= \text{im} \begin{pmatrix} I \\ H_G^+ \end{pmatrix} + \text{im} \begin{pmatrix} I \\ T_G \end{pmatrix}. \end{aligned}$$

Here H_G^+ denotes the Hankel operator $\Pi_+ M_G|_{\mathcal{L}_2(\mathbb{R}^-)}: \mathcal{L}_2(\mathbb{R}^-) \rightarrow \mathcal{L}_2(\mathbb{R}^+)$. Since (A, B) is reachable, $\text{im } H_G^+ = C e^{At} \mathbb{1}(t) \mathbb{R}^n$. This proves (5.4). From this we conclude that for the two input-output behaviors $\mathfrak{B}_i = \text{im} \begin{pmatrix} I \\ G_i(\frac{d}{dt}) \end{pmatrix}$ we have $\delta'(G_1, G_2) = d_{SB}(\mathfrak{B}_1, \mathfrak{B}_2)$ as defined by Definition 2.3.

We now turn to the problem of computing for two given behaviors their distance in the SB-metric. It turns out that not much work needs to be done for this, since the SB-metric is in a sense *dual* to the Z-metric. We first prove the following lemma.

LEMMA 5.6. *Let $\mathfrak{B} \in \mathcal{L}_{\text{cont}}^q$. The orthogonal projection of $\mathfrak{B} \cap \mathcal{L}_2(\mathbb{R})$ onto $\mathcal{L}_2(\mathbb{R}^+)$ is equal to the orthogonal complement in $\mathcal{L}_2(\mathbb{R}^+)$ of $\mathfrak{B}^* \cap \mathcal{L}_2(\mathbb{R}^+)$:*

$$\Pi_+(\mathfrak{B} \cap \mathcal{L}_2(\mathbb{R})) = (\mathfrak{B}^* \cap \mathcal{L}_2(\mathbb{R}^+))^\perp \cap \mathcal{L}_2(\mathbb{R}^+).$$

Proof. (⊂) Let $w_+ \in \Pi_+(\mathfrak{B} \cap \mathcal{L}_2(\mathbb{R}))$, and let $w_+ = \Pi_+ w$ with $w \in \mathfrak{B} \cap \mathcal{L}_2(\mathbb{R})$. Take any $v \in \mathfrak{B}^* \cap \mathcal{L}_2(\mathbb{R}^+)$. Since $\mathcal{L}_2(\mathbb{R}^+) \subset \mathcal{L}_2(\mathbb{R})$, by Lemma 4.5 we have $v \in (\mathfrak{B} \cap \mathcal{L}_2(\mathbb{R}))^\perp$. Thus $\int_{-\infty}^\infty v^\top w_+ dt = \int_0^\infty v^\top w_+ dt = \int_0^\infty v^\top w dt = \int_{-\infty}^\infty v^\top w dt = 0$. Thus $w_+ \in (\mathfrak{B}^* \cap \mathcal{L}_2(\mathbb{R}^+))^\perp$.

(⊃) First note that $\Pi_+(\mathfrak{B} \cap \mathcal{L}_2(\mathbb{R})) = \Pi_+(\mathfrak{B}^* \cap \mathcal{L}_2(\mathbb{R}))^\perp$. Since $\Pi_+ = \Pi_+^*$, the latter equals $(\Pi_+^{-1}(\mathfrak{B}^* \cap \mathcal{L}_2(\mathbb{R})))^\perp$, the orthogonal complement of the inverse image under Π_+ of $\mathfrak{B}^* \cap \mathcal{L}_2(\mathbb{R})$. Now let $w \in (\mathfrak{B}^* \cap \mathcal{L}_2(\mathbb{R}^+))^\perp \cap \mathcal{L}_2(\mathbb{R}^+)$. Take any $v \in \Pi_+^{-1}(\mathfrak{B}^* \cap \mathcal{L}_2(\mathbb{R}))$. Then $v_+ := \Pi_+ v \in \mathfrak{B}^* \cap \mathcal{L}_2(\mathbb{R}^+)$. Thus $\int_{-\infty}^\infty w^\top v_+ dt = 0$. Therefore, $\int_{-\infty}^\infty w^\top v dt = \int_0^\infty w^\top v dt = \int_0^\infty w^\top v_+ dt = \int_{-\infty}^\infty w^\top v_+ dt = 0$. We conclude that $w \in (\Pi_+^{-1}(\mathfrak{B}^* \cap \mathcal{L}_2(\mathbb{R})))^\perp = \Pi_+(\mathfrak{B} \cap \mathcal{L}_2(\mathbb{R}))$. This completes the proof of the lemma. \square

By applying this lemma, we obtain the following theorem that expresses the distance between two behaviors in the SB-metric in terms of the distance of the dual behaviors in the Z-metric.

THEOREM 5.7. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$. Then $d_{SB}(\mathfrak{B}_1, \mathfrak{B}_2) = d_Z(\mathfrak{B}_1^*, \mathfrak{B}_2^*)$.*

Proof. By Lemma 5.6 we have

$$(5.5) \quad d_{SB}(\mathfrak{B}_1, \mathfrak{B}_2) = \text{gap}_{\mathcal{L}_2}((\mathfrak{B}_1^* \cap \mathcal{L}_2(\mathbb{R}^+))^\perp \cap \mathcal{L}_2(\mathbb{R}^+), (\mathfrak{B}_2^* \cap \mathcal{L}_2(\mathbb{R}^+))^\perp \cap \mathcal{L}_2(\mathbb{R}^+)).$$

For $i = 1, 2$, let Π_i be the orthogonal projection of $\mathcal{L}_2(\mathbb{R})$ onto $\mathfrak{B}_i^* \cap \mathcal{L}_2(\mathbb{R}^+)$. Then $I - \Pi_i$ is the orthogonal projection onto $(\mathfrak{B}_i^* \cap \mathcal{L}_2(\mathbb{R}^+))^\perp$. As before let Π_+ be the orthogonal projection onto $\mathcal{L}_2(\mathbb{R}^+)$. Clearly $\Pi_i \Pi_+ = \Pi_+ \Pi_i = \Pi_i$. It is easily verified that $((I - \Pi_i) \Pi_+)^2 = (I - \Pi_i) \Pi_+$, that

$$(5.6) \quad \text{im}((I - \Pi_i) \Pi_+) = (\mathfrak{B}_i^* \cap \mathcal{L}_2(\mathbb{R}^+))^\perp \cap \mathcal{L}_2(\mathbb{R}^+),$$

and that $(I - \Pi_i) \Pi_+$ is self-adjoint. Hence $(I - \Pi_i) \Pi_+$ is in fact the orthogonal projection onto the subspace (5.6). As a consequence we find that (5.5) is equal to

$$\|(I - \Pi_1) \Pi_+ - (I - \Pi_2) \Pi_+\| = \|\Pi_1 - \Pi_2\| = \text{gap}_{\mathcal{L}_2}(\mathfrak{B}_1^* \cap \mathcal{L}_2(\mathbb{R}^+), \mathfrak{B}_2^* \cap \mathcal{L}_2(\mathbb{R}^+)),$$

which by Definition 2.2 equals $d_Z(\mathfrak{B}_1^*, \mathfrak{B}_2^*)$. This completes the proof. \square

As a consequence, for given controllable behaviors the distance in the SB-metric can be computed by computing the distance between the dual behaviors in the Z-metric. Again, this involves the solutions of two \mathcal{H}_∞ optimization problems.

THEOREM 5.8. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$, $\mathfrak{m}(\mathfrak{B}_1) = \mathfrak{m}(\mathfrak{B}_2)$. Let $\tilde{G}_1, \tilde{G}_2 \in R\mathcal{H}_\infty^-$ be such that $\mathfrak{B}_1 = \ker \tilde{G}_1(\frac{d}{dt})$ and $\mathfrak{B}_2 = \ker \tilde{G}_2(\frac{d}{dt})$ with \tilde{G}_1 and \tilde{G}_2 co-inner and left-prime over $R\mathcal{H}_\infty^-$. Then we have*

$$(5.7) \quad d_{SB}(\mathfrak{B}_1, \mathfrak{B}_2) = \max \left\{ \inf_{Q \in R\mathcal{H}_\infty} \|\tilde{G}_1^* - \tilde{G}_2^* Q\|_\infty, \inf_{Q \in R\mathcal{H}_\infty} \|\tilde{G}_2^* - \tilde{G}_1^* Q\|_\infty \right\}.$$

Proof. Note that $\mathfrak{B}_1^* = \text{im} \tilde{G}_1^*(\frac{d}{dt})$ and $\mathfrak{B}_2^* = \text{im} \tilde{G}_2^*(\frac{d}{dt})$, that $\mathfrak{m}(\mathfrak{B}_1^*) = \mathfrak{m}(\mathfrak{B}_2^*)$, and that $\tilde{G}_1^*, \tilde{G}_2^* \in R\mathcal{H}_\infty$ are inner and right prime over $R\mathcal{H}_\infty$. The result then follows by applying Proposition 5.3. \square

Remark 5.9. According to Theorem 7 in [12], for the special case of stable input-state-output systems the concept of distance between behaviors that was introduced in [12] coincides with the SB-metric defined in our paper. Most likely, the distance concept from [12] in fact coincides with the SB-metric for general controllable behaviors. This issue is left for future research.

6. Vinnicombe metric. In [21], [22], Vinnicombe proposed a notion of distance between transfer matrices in the input-output framework often referred to as the ν -gap (see also [2], [14]). The main difference between the ν -gap and both the \mathcal{L}_2 -gap and the usual gap metric studied in [7] is that the ν -gap does not have an apparent, direct interpretation in terms of “gap between subspaces” of the Hilbert space $\mathcal{L}_2(\mathbb{R})$. Instead, in computing the value of the ν -gap between two transfer matrices, an important role is played by the winding number of a rational matrix associated with the given transfer matrices.

In the present section we will generalize the notion of ν -gap and introduce a metric on the set of controllable behaviors with the same input cardinality. This will yield a *representation-free* characterization of the ν -gap between two systems.

DEFINITION 6.1. Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$, and $\mathfrak{m}(\mathfrak{B}_1) = \mathfrak{m}(\mathfrak{B}_2)$. Let $G_1, G_2 \in R\mathcal{H}_\infty$ be inner and right prime (over $R\mathcal{H}_\infty$) such that $\mathfrak{B}_1 = \text{im } G_1(\frac{d}{dt})$ and $\mathfrak{B}_2 = \text{im } G_2(\frac{d}{dt})$. We define the Vinnicombe metric $d_V(\mathfrak{B}_1, \mathfrak{B}_2)$ by

$$(6.1) \quad d_V(\mathfrak{B}_1, \mathfrak{B}_2) := \begin{cases} \text{gap}_{\mathcal{L}_2}(\text{im } M_{G_1}, \text{im } M_{G_2}) & \text{if } \det(G_2^* G_1)(i\omega) \neq 0 \ \forall \omega \in \mathbb{R} \\ & \text{and } \text{wno } \det(G_2^* G_1) = 0, \\ 1 & \text{otherwise.} \end{cases}$$

It should of course be checked whether this definition is correct, in the sense that the definition of $d_V(\mathfrak{B}_1, \mathfrak{B}_2)$ is independent of the rational matrices G_1, G_2 . For this, we prove the following lemma.

LEMMA 6.2. Let $G, G' \in R\mathcal{H}_\infty$ be inner and right prime. Then $\text{im } G(\frac{d}{dt}) = \text{im } G'(\frac{d}{dt})$ if and only if there exists a constant unitary matrix U such that $G' = GU$.

Proof. From [15], $\text{im } G(\frac{d}{dt}) = \text{im } G'(\frac{d}{dt})$ if and only if there exists a nonsingular rational matrix U such that $G' = GU$. Let $G^+, G'^+ \in R\mathcal{H}_\infty$ be left inverses of G and G' , respectively. Then $U = G^+ G'$, so $U \in R\mathcal{H}_\infty$. Also, $U^{-1} = G'^+ G$, so $U^{-1} \in R\mathcal{H}_\infty$. Finally, note that $I = G'^+ G' = U^* G^* G U = U^* U$, so $U^{-1} = U^*$. Since $U^* \in R\mathcal{H}_\infty^-$ we conclude that U is constant. \square

To prove that Definition 6.1 is correct, let $G'_1, G'_2 \in R\mathcal{H}_\infty$ be alternative rational matrices, both inner and right prime, such that $\mathfrak{B}_1 = \text{im } G'_1(\frac{d}{dt})$ and $\mathfrak{B}_2 = \text{im } G'_2(\frac{d}{dt})$. Obviously, $\text{im } M_{G_i} = \text{im } M_{G'_i} = \mathfrak{B}_i \cap \mathcal{L}_2(\mathbb{R})$. Also, from the previous lemma we have that $G'_i = G_i U_i$ for constant unitary matrices U_i . Thus, $G_2^* G'_1 = U_2^* G_2^* G_1 U_1$, so $\det(G_2^* G_1)(i\omega) \neq 0$ for all $\omega \in \mathbb{R}$ if and only if $\det(G_2'^* G'_1)(i\omega) \neq 0$ for all $\omega \in \mathbb{R}$. Also, $\text{wno } \det(G_2^* G_1) = 0$ if and only if $\text{wno } \det(G_2'^* G'_1) = 0$.

A proof of the fact that $d_V(\mathfrak{B}_1, \mathfrak{B}_2)$ as defined above indeed defines a metric (on the subset of $\mathcal{L}_{\text{cont}}^q$ of all controllable behaviors with the same input cardinality) can be given by adapting the corresponding proof in the input-output setting. For this we refer to [21]. As shorthand terminology, in what follows we will refer to this metric as the *V-metric*.

Of course, computing the gap between two controllable behaviors in the Vinnicombe metric only involves checking an appropriate winding number, possibly followed by a computation of the gap in the \mathcal{L}_2 -metric. The following result follows immediately from Theorem 5.1.

THEOREM 6.3. Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$, and $\mathfrak{m}(\mathfrak{B}_1) = \mathfrak{m}(\mathfrak{B}_2)$. Let $G_1, G_2 \in R\mathcal{H}_\infty$ be inner and right prime (over $R\mathcal{H}_\infty$) such that $\mathfrak{B}_1 = \text{im } G_1(\frac{d}{dt})$ and $\mathfrak{B}_2 = \text{im } G_2(\frac{d}{dt})$. Also, let $\tilde{G}_1, \tilde{G}_2 \in R\mathcal{L}_\infty$ such that $\mathfrak{B}_1 = \ker \tilde{G}_1(\frac{d}{dt})$ and $\mathfrak{B}_2 = \ker \tilde{G}_2(\frac{d}{dt})$ with \tilde{G}_1, \tilde{G}_2 co-inner. Then

$$d_V(\mathfrak{B}_1, \mathfrak{B}_2) := \begin{cases} \|\tilde{G}_2 G_1\|_\infty (= \|\tilde{G}_1 G_2\|_\infty) & \text{if } \det(G_2^* G_1)(i\omega) \neq 0 \ \forall \omega \in \mathbb{R}, \\ & \text{and } \text{wno } \det(G_2^* G_1) = 0, \\ 1 & \text{otherwise.} \end{cases}$$

The original definition of V-metric as given in [21], as well as its generalization given above, are not entirely satisfactory, since they are given in terms of the rational matrices representing the systems. In the remainder of this section, we will instead establish a *representation-free* characterization of the distance between two controllable behaviors in the V-metric, no longer using the matrices appearing in their rational representations. Before doing this, we first introduce some additional material on linear differential systems.

6.1. More about behaviors. In section 2 we introduced linear differential systems as those systems whose behavior can be represented as the kernel of a polynomial differential operator, $\mathfrak{B} = \ker R(\frac{d}{dt})$, with R a real polynomial matrix. Another representation is a *latent variable representation*, defined through polynomial matrices R and M by $R(\frac{d}{dt})w = M(\frac{d}{dt})v$ with $\mathfrak{B} = \{w \in \mathcal{L}_{\text{loc}} \mid \exists v \in \mathcal{L}_{\text{loc}} \text{ such that } R(\frac{d}{dt})w = M(\frac{d}{dt})v\}$. The variable v is called a latent variable. If the latent variable has the property of state (see [11], [18]), then the latent variable is called a *state variable*, and the latent variable representation is called a *state representation* of \mathfrak{B} . For any $\mathfrak{B} \in \mathcal{L}^q$ many state representations exist, but the *minimal number of components* of the state variable in any state representation of \mathfrak{B} is an invariant for \mathfrak{B} . This nonnegative integer is called *the McMillan degree* of \mathfrak{B} , denoted by $\mathfrak{n}(\mathfrak{B})$. It is a basic fact that the McMillan degree of \mathfrak{B} and its dual \mathfrak{B}^* are the same: $\mathfrak{n}(\mathfrak{B}) = \mathfrak{n}(\mathfrak{B}^*)$. The following lemma expresses the McMillan degree in terms of rational representations.

LEMMA 6.4. *Let G be a proper real rational matrix. Then the following holds:*

1. *If G is left prime over $\mathbb{R}(\xi)_P$ and $G = P^{-1}Q$ is a left coprime factorization over $\mathbb{R}[\xi]$, then the McMillan degree of the behavior $\ker G(\frac{d}{dt})$ is equal to $\deg \det(P)$.*
2. *If G has no zeros and is right prime over $\mathbb{R}(\xi)_P$ and $G = MN^{-1}$ is a right coprime factorization over $\mathbb{R}[\xi]$, then the McMillan degree of the behavior $\text{im } G(\frac{d}{dt})$ is equal to $\deg \det(N)$.*

Proof. 1. The crux is that if Q is a full row rank polynomial matrix with p rows, then the McMillan degree of $\ker Q(\frac{d}{dt})$ is equal to the maximum of the degrees of the determinants over all $p \times p$ minors of Q (see [11]). Now let G have p rows and be left prime over $\mathbb{R}(\xi)_P$. Then by [27, p. 240], it has a biproper $p \times p$ minor, say, \tilde{G} . The corresponding $p \times p$ minor of Q , say, \tilde{Q} , satisfies $\tilde{G} = P^{-1}\tilde{Q}$. In addition, for every $p \times p$ minor \tilde{Q}' , $P^{-1}\tilde{Q}'$ is proper. Thus for every minor \tilde{Q}' we have $\deg \det(\tilde{Q}') \leq \deg \det(P)$, while $\deg \det(\tilde{Q}) = \deg \det(P)$. This proves the claim.

2. This is proved along the same lines, using the fact that if M is a full column rank polynomial matrix with m columns, having no zeros, then the McMillan degree of $\text{im } M(\frac{d}{dt})$ is equal to the maximum of the degrees of the determinants over all $m \times m$ minors of M . \square

Next we will briefly discuss autonomous behaviors; see [11, p. 66]. Let $\mathfrak{B} \in \mathcal{L}^q$. We call the behavior *autonomous* if it has no input variables, i.e., if $\mathfrak{m}(\mathfrak{B}) = 0$. Being autonomous is reflected in kernel representation as follows: \mathfrak{B} is autonomous if and only if there exists a square, nonsingular polynomial matrix R such that $\mathfrak{B} = \ker R(\frac{d}{dt})$. Also, a given $\mathfrak{B} \in \mathcal{L}^q$ is autonomous if and only if it is a finite-dimensional subspace of $\mathcal{L}_{\text{loc}}(\mathbb{R}, \mathbb{R}^q)$. In fact, its dimension is then equal to the degree of the polynomial $\det(R)$. Also, this number is equal to the McMillan degree of \mathfrak{B} , i.e., $\dim \mathfrak{B} = \mathfrak{n}(\mathfrak{B})$.

The roots of the polynomial $\det(R)$ are called the *frequencies* of \mathfrak{B} . They only depend on \mathfrak{B} , since if $\mathfrak{B} = \ker R'(\frac{d}{dt})$ is a second kernel representation of \mathfrak{B} with R' square and nonsingular we must have $R' = UR$ for some unimodular polynomial matrix U . A frequency λ with $\text{Re}(\lambda) = 0$, i.e., λ lies on the imaginary axis, is called an *imaginary frequency*. The following is easily seen, and we omit the proof.

LEMMA 6.5. *Let $\mathfrak{B} \in \mathcal{L}^q$. Let M be a polynomial matrix with q columns. Then \mathfrak{B} is autonomous if and only if $M(\frac{d}{dt})\mathfrak{B}$ is autonomous. Let $\lambda \in \mathbb{C}$ be such that $M(\lambda)$ has full column rank. Then λ is a frequency of \mathfrak{B} if and only if λ is a frequency of $M(\frac{d}{dt})\mathfrak{B}$.*

If \mathfrak{B} is autonomous and has no imaginary frequencies, then $\mathfrak{B} = \mathfrak{B}_{\text{stab}} \oplus \mathfrak{B}_{\text{anti}}$ uniquely, with $\mathfrak{B}_{\text{stab}}$ stable (i.e., $\lim_{t \rightarrow \infty} w(t) = 0$ for all $w \in \mathfrak{B}_{\text{stab}}$) and $\mathfrak{B}_{\text{anti}}$ anti-stable (i.e., $\lim_{t \rightarrow -\infty} w(t) = 0$ for all $w \in \mathfrak{B}_{\text{anti}}$).

6.2. A representation-free approach to the Vinnicombe metric. In this subsection we present a representation-free approach to the Vinnicombe metric. We first prove a lemma that expresses the winding number appearing in the definition of the Vinnicombe metric in terms of McMillan degrees associated with the underlying behaviors.

LEMMA 6.6. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$ with $\mathfrak{m}(\mathfrak{B}_1) = \mathfrak{m}(\mathfrak{B}_2)$. Let $G_1, G_2 \in R\mathcal{L}_{\infty}$ such that $\mathfrak{B}_i = \text{im } G_i(\frac{d}{dt})$. Then*

1. $G_2^* G_1$ is a nonsingular rational matrix if and only if $\mathfrak{B}_1 \cap \mathfrak{B}_2^*$ is autonomous,
2. $(G_2^* G_1)(i\omega)$ is nonsingular for all $\omega \in \mathbb{R}$ if and only if $\mathfrak{B}_1 \cap \mathfrak{B}_2^*$ is autonomous and has no imaginary frequencies.

Furthermore, if $G_1, G_2 \in R\mathcal{H}_{\infty}$, G_2 is left prime over $R\mathcal{L}_{\infty}$ and 2 above holds, then

$$\text{wno det}(G_2^* G_1) = \mathfrak{n}(\mathfrak{B}_2) - \dim(\mathfrak{B}_1 \cap \mathfrak{B}_2^*)_{\text{anti}},$$

where $(\mathfrak{B}_1 \cap \mathfrak{B}_2^*)_{\text{anti}}$ denotes the antistable part of $\mathfrak{B}_1 \cap \mathfrak{B}_2^*$.

Proof. Let $G_1 = MN^{-1}$ be a right coprime factorization over $\mathbb{R}[\xi]$ and $G_2^* = P^{-1}Q$ a left coprime factorization over $\mathbb{R}[\xi]$. Then $\mathfrak{B}_1 = \text{im } M(\frac{d}{dt})$. Furthermore, since $\mathfrak{B}_2^* = \ker G_2^*(\frac{d}{dt})$, we have $\mathfrak{B}_2^* = \ker Q(\frac{d}{dt})$. It is then easily verified that $\mathfrak{B}_1 \cap \mathfrak{B}_2^* = M(\frac{d}{dt}) \ker(QM)(\frac{d}{dt})$.

1. Now $G_2^* G_1 = P^{-1}QM N^{-1}$, hence $G_2^* G_1$ is nonsingular if and only if QM is nonsingular, and equivalently, $\mathfrak{B}_1 \cap \mathfrak{B}_2^*$ is autonomous. Statement 2 then follows from Lemma 6.5. Assume now that $G_1, G_2 \in R\mathcal{H}_{\infty}$, G_2 is left prime over $R\mathcal{L}_{\infty}$, and condition 2 holds. We have

$$\det(G_2^* G_1) = \frac{\det(QM)}{\det(P)\det(N)}.$$

The winding number $\text{wno det}(G_2^* G_1)$ is equal to number of roots of $\det(QM)$ in \mathbb{C}^+ minus the number of roots of the product $\det(P)\det(N)$ in \mathbb{C}^+ . The number of roots of $\det(QM)$ in \mathbb{C}^+ is equal to $\dim(\mathfrak{B}_1 \cap \mathfrak{B}_2^*)_{\text{anti}}$, while the number of roots of $\det(P)\det(N)$ in \mathbb{C}^+ is equal to the degree of $\det(P)$. (Note that N is Hurwitz, and P is anti-Hurwitz.) Finally, from left-primeness of G_2 , by Lemma 6.4 the degree of $\det(P)$ is equal to the McMillan degree of \mathfrak{B}_2^* , which is equal to the McMillan degree of \mathfrak{B}_2 . This completes the proof. \square

This immediately leads to the following result, which expresses the distance in the V-metric between two given behaviors completely in terms of the behaviors and no longer in terms of their rational representations.

THEOREM 6.7. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$, $\mathfrak{m}(\mathfrak{B}_1) = \mathfrak{m}(\mathfrak{B}_2)$. Then*

$$d_V(\mathfrak{B}_1, \mathfrak{B}_2) = \begin{cases} d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) & \text{if } \mathfrak{B}_1 \cap \mathfrak{B}_2^* \text{ autonomous, has no imaginary} \\ & \text{frequencies, and } \dim(\mathfrak{B}_1 \cap \mathfrak{B}_2^*)_{\text{anti}} = \mathfrak{n}(\mathfrak{B}_2), \\ 1 & \text{otherwise.} \end{cases}$$

We conclude this subsection with establishing some basic properties of the V-metric. In contrast to the \mathcal{L}_2 -metric, the V-metric is not invariant under dualization. The following result gives conditions under which invariance does hold.

THEOREM 6.8. Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$, $\mathfrak{m}(\mathfrak{B}_1) = \mathfrak{m}(\mathfrak{B}_2)$. Then the following hold:

1. If $\mathfrak{n}(\mathfrak{B}_1) = \mathfrak{n}(\mathfrak{B}_2)$, then $d_V(\mathfrak{B}_1, \mathfrak{B}_2) = d_V(\mathfrak{B}_1^*, \mathfrak{B}_2^*)$.
2. If $d_V(\mathfrak{B}_1, \mathfrak{B}_2) < 1$ and $d_V(\mathfrak{B}_1^*, \mathfrak{B}_2^*) < 1$, then $d_V(\mathfrak{B}_1, \mathfrak{B}_2) = d_V(\mathfrak{B}_1^*, \mathfrak{B}_2^*)$ if and only if $\mathfrak{n}(\mathfrak{B}_1) = \mathfrak{n}(\mathfrak{B}_2)$.

Proof. 1. Consider the following conditions: $\mathfrak{B}_1 \cap \mathfrak{B}_2^*$ is autonomous and has no imaginary frequencies and $\dim(\mathfrak{B}_1 \cap \mathfrak{B}_2^*)_{\text{anti}} = \mathfrak{n}(\mathfrak{B}_2)$. Obviously, since $(\mathfrak{B}_1^*)^* = \mathfrak{B}_1$, and by the assumption that $\mathfrak{n}(\mathfrak{B}_2) = \mathfrak{n}(\mathfrak{B}_1) = \mathfrak{n}(\mathfrak{B}_1^*)$, this set of conditions is equivalent to the following: $\mathfrak{B}_2^* \cap (\mathfrak{B}_1^*)^*$ is autonomous, has no imaginary frequencies, and $\dim(\mathfrak{B}_2^* \cap (\mathfrak{B}_1^*)^*)_{\text{anti}} = \mathfrak{n}(\mathfrak{B}_1^*)$. Now we distinguish between two cases: (a) the above equivalent sets of conditions hold. Then $d_V(\mathfrak{B}_1, \mathfrak{B}_2) = d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2)$ and $d_V(\mathfrak{B}_2^*, \mathfrak{B}_1^*) = d_{\mathcal{L}_2}(\mathfrak{B}_2^*, \mathfrak{B}_1^*)$. By symmetry we then obtain $d_V(\mathfrak{B}_1, \mathfrak{B}_2) = d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) = d_{\mathcal{L}_2}(\mathfrak{B}_2, \mathfrak{B}_1) = d_{\mathcal{L}_2}(\mathfrak{B}_2^*, \mathfrak{B}_1^*) = d_V(\mathfrak{B}_2^*, \mathfrak{B}_1^*) = d_V(\mathfrak{B}_1^*, \mathfrak{B}_2^*)$. (b) The conditions do not hold. In that case both $d_V(\mathfrak{B}_1, \mathfrak{B}_2) = 1$ and $d_V(\mathfrak{B}_2^*, \mathfrak{B}_1^*) = 1$, so again $d_V(\mathfrak{B}_1, \mathfrak{B}_2) = d_V(\mathfrak{B}_1^*, \mathfrak{B}_2^*)$.

2. From $d_V(\mathfrak{B}_1, \mathfrak{B}_2) < 1$ it follows that $\dim(\mathfrak{B}_1 \cap \mathfrak{B}_2^*)_{\text{anti}} = \mathfrak{n}(\mathfrak{B}_2)$. On the other hand, $d_V(\mathfrak{B}_2^*, \mathfrak{B}_1^*) < 1$ implies that $\dim(\mathfrak{B}_2^* \cap (\mathfrak{B}_1^*)^*)_{\text{anti}} = \mathfrak{n}(\mathfrak{B}_1^*)$. Thus, $\mathfrak{n}(\mathfrak{B}_2) = \mathfrak{n}(\mathfrak{B}_1^*) = \mathfrak{n}(\mathfrak{B}_1)$. \square

Example 6.9. We give an example in which dualization changes the values of the V-metric if the McMillan degrees of the behaviors are not equal. Define $\mathfrak{B}_i := \text{im } G_i(\frac{d}{dt})$ with

$$G_1(\xi) := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad G_2(\xi) := \frac{1}{\xi + \sqrt{5}} \begin{pmatrix} 1 \\ \xi - 2 \end{pmatrix}.$$

Note that $\mathfrak{n}(\mathfrak{B}_1) = 0$ while $\mathfrak{n}(\mathfrak{B}_2) = 1$. We compute $(G_2^* G_1)(\xi) = -\frac{\xi+3}{\sqrt{2}(\xi-\sqrt{5})}$. Clearly its winding number is unequal to 0, so we have $d_V(\mathfrak{B}_1, \mathfrak{B}_2) = 1$. Now, $\mathfrak{B}_1^* = \ker G_1^*(\frac{d}{dt}) = \text{im } H_1(\frac{d}{dt})$ and $\mathfrak{B}_2^* = \ker G_2^*(\frac{d}{dt}) = \text{im } \tilde{H}_2(\frac{d}{dt}) = \text{im } H_2(\frac{d}{dt})$, where

$$H_1(\xi) := \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \tilde{H}_2(\xi) := -\frac{1}{\xi + \sqrt{5}} \begin{pmatrix} 1 & -\xi - 2 \end{pmatrix}, \quad H_2(\xi) := \frac{1}{\xi + \sqrt{5}} \begin{pmatrix} \xi + 2 \\ 1 \end{pmatrix}.$$

We compute $(H_2^* H_1)(\xi) = \frac{\xi-1}{\sqrt{2}(\xi-\sqrt{5})}$. Its winding number is equal to 0, so $d_V(\mathfrak{B}_1^*, \mathfrak{B}_2^*) = \|\tilde{H}_2 H_1\|_\infty$. Now, $(\tilde{H}_2 H_1)(\xi) = \frac{\xi+3}{\sqrt{2}(\xi-\sqrt{5})}$, which indeed yields $\|\tilde{H}_2 H_1\|_\infty = \frac{3}{\sqrt{10}} < 1$.

We conclude this subsection by stating a result that was proved in [21] in an input-output setting and that expresses the computation of the distance between two controllable behaviors in the V-metric as an optimization problem.

PROPOSITION 6.10. Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$, $\mathfrak{m}(\mathfrak{B}_1) = \mathfrak{m}(\mathfrak{B}_2)$. Let $G_1, G_2 \in R\mathcal{H}_\infty$ be inner and right prime (over $R\mathcal{H}_\infty$) such that $\mathfrak{B}_1 = \text{im } G_1(\frac{d}{dt})$ and $\mathfrak{B}_2 = \text{im } G_2(\frac{d}{dt})$. Then

$$d_V(\mathfrak{B}_1, \mathfrak{B}_2) = \inf_{Q, Q^{-1} \in R\mathcal{L}_\infty, \text{wno det}(Q)=0} \|G_1 - G_2 Q\|_\infty.$$

Recall that computation of the Z-metric was formulated in an analogous way in Proposition 5.4. Again, the proof given in [21] carries over to our framework and is omitted here.

Remark 6.11. Although in Theorem 6.7 we established a representation-free characterization of the V-metric, there still remains the question of whether this metric can be given a “gap in the Hilbert space” interpretation like the Z-metric and the SB-metric, for example, by intersecting the behaviors with some “natural” subspace of

$\mathcal{L}_2(\mathbb{R})$ or by applying a suitable projection. This question remains unanswered and is left for future research.

7. Comparison of the metrics. In this section we will compare the metrics that we introduced in sections 2 and 6. It will turn out that the \mathcal{L}_2 -gap is dominated by the V-gap, which in turn is dominated by the Z-gap. The \mathcal{L}_2 -gap is also dominated by the SB-gap. However, the SB-gap will in general turn out to be incomparable with both the V-gap as well as the Z-gap. We will also compare the topologies induced by the metrics. Generalizing a result from [21], we will find that the topologies induced by the V-metric and the Z-metric coincide. We will also show that if we restrict the V-metric and the SB-metric to the subset $\mathcal{L}_{\text{cont}}^q(n)$ of all controllable behaviors of fixed McMillan degree n , then they induce the same topology on that subset. This new result will generalize a result from [13] on stable input-output systems.

Our first proposition is a simple generalization of results from [21].

PROPOSITION 7.1. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$, $\mathfrak{m}(\mathfrak{B}_1) = \mathfrak{m}(\mathfrak{B}_2)$. Then*

$$d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) \leq d_V(\mathfrak{B}_1, \mathfrak{B}_2) \leq d_Z(\mathfrak{B}_1, \mathfrak{B}_2).$$

Proof. The inequality between $d_{\mathcal{L}_2}$ and d_V follows immediately from Theorem 6.7. The one between d_V and d_Z follows by combining Propositions 5.4 and 6.10. \square

Next, we study the question of how the SB-metric relates to the other metrics. We first compare with the \mathcal{L}_2 -metric and the V-metric.

THEOREM 7.2. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$, $\mathfrak{m}(\mathfrak{B}_1) = \mathfrak{m}(\mathfrak{B}_2)$. Then the following hold:*

1. $d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) \leq d_{SB}(\mathfrak{B}_1, \mathfrak{B}_2)$.
2. *If $\mathfrak{n}(\mathfrak{B}_1) = \mathfrak{n}(\mathfrak{B}_2)$, then $d_V(\mathfrak{B}_1, \mathfrak{B}_2) \leq d_{SB}(\mathfrak{B}_1, \mathfrak{B}_2)$.*

Proof. 1. $d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) = d_{\mathcal{L}_2}(\mathfrak{B}_1^*, \mathfrak{B}_2^*) \leq d_Z(\mathfrak{B}_1^*, \mathfrak{B}_2^*) = d_{SB}(\mathfrak{B}_1, \mathfrak{B}_2)$. The first equality follows from the fact that the \mathcal{L}_2 -metric is invariant under dualization, the second follows from Proposition 7.1, and the third follows from Theorem 5.7. 2. By Theorem 6.8, if $\mathfrak{n}(\mathfrak{B}_1) = \mathfrak{n}(\mathfrak{B}_2)$, then $d_V(\mathfrak{B}_1, \mathfrak{B}_2) = d_V(\mathfrak{B}_1^*, \mathfrak{B}_2^*)$. Next, by Proposition 7.1 and Theorem 5.7, respectively, $d_V(\mathfrak{B}_1^*, \mathfrak{B}_2^*) \leq d_Z(\mathfrak{B}_1^*, \mathfrak{B}_2^*) = d_{SB}(\mathfrak{B}_1, \mathfrak{B}_2)$. \square

Remark 7.3. According to the previous theorem, on every set $\mathcal{L}_{\text{cont}}^q(n)$ consisting of all controllable behaviors with fixed McMillan degree n , the V-metric is dominated by the SB-metric. In general these two metrics turn out to be incomparable. If for two given behaviors we have $d_V(\mathfrak{B}_1, \mathfrak{B}_2) < 1$, then of course $d_V(\mathfrak{B}_1, \mathfrak{B}_2) = d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) \leq d_{SB}(\mathfrak{B}_1, \mathfrak{B}_2)$. However, in the following example we present two behaviors such that $d_{SB}(\mathfrak{B}_1, \mathfrak{B}_2) < d_V(\mathfrak{B}_1, \mathfrak{B}_2)$.

Example 7.4. Consider the behaviors \mathfrak{B}_1 and \mathfrak{B}_2 from Example 6.9. We computed that $d_V(\mathfrak{B}_1, \mathfrak{B}_2) = 1$ and $d_V(\mathfrak{B}_1^*, \mathfrak{B}_2^*) = \frac{3}{\sqrt{10}}$. It was shown in [21, p. 241] that for \mathfrak{B}_1 and \mathfrak{B}_2 with input cardinality equal to 1, their distance in the V-metric coincides with that in the Z-metric. Thus $d_Z(\mathfrak{B}_1^*, \mathfrak{B}_2^*) = \frac{3}{\sqrt{10}}$. This implies that $d_{SB}(\mathfrak{B}_1, \mathfrak{B}_2) = \frac{3}{\sqrt{10}} < 1$. Finally, we compare the SB-metric with the Z-metric. By Theorem 5.7 $d_Z(\mathfrak{B}_1^*, \mathfrak{B}_2^*) = d_{SB}(\mathfrak{B}_1, \mathfrak{B}_2)$. Therefore these two metrics are again incomparable in the sense that $d_Z \not\leq d_{SB}$ nor $d_Z \not\geq d_{SB}$. In fact, for every pair of behaviors $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$ we have $d_Z(\mathfrak{B}_1, \mathfrak{B}_2) < d_{SB}(\mathfrak{B}_1, \mathfrak{B}_2)$ if and only if $d_Z(\mathfrak{B}_1^*, \mathfrak{B}_2^*) > d_{SB}(\mathfrak{B}_1^*, \mathfrak{B}_2^*)$.

Next, we will turn to a comparison of the *topologies* induced by the metrics. It follows from the inequalities given in this section that the topology induced by the \mathcal{L}_2 -metric is coarser than those induced by the others. In fact, this topology can be shown to be *strictly* coarser than the other topologies, and it was argued in [21] that, due to this fact, it is in general not useful in robust control.

By generalizing Theorem IV.4 in [21], it can be shown that for any pair $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$ there exists a constant $0 < c \leq 1$ (depending on \mathfrak{B}_1) such that

$$(7.1) \quad c d_Z(\mathfrak{B}_1, \mathfrak{B}_2) \leq d_V(\mathfrak{B}_1, \mathfrak{B}_2) \leq d_Z(\mathfrak{B}_1, \mathfrak{B}_2).$$

Obviously, this inequality implies that the topologies induced by the Z-metric and the V-metric coincide. We will now compare the topologies of the SB-metric and the V-metric.

THEOREM 7.5. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$, $\mathfrak{m}(\mathfrak{B}_1) = \mathfrak{m}(\mathfrak{B}_2)$. Assume that $\mathfrak{n}(\mathfrak{B}_1) = \mathfrak{n}(\mathfrak{B}_2)$. Then there exists $0 < c \leq 1$ such that*

$$c d_{SB}(\mathfrak{B}_1, \mathfrak{B}_2) \leq d_V(\mathfrak{B}_1, \mathfrak{B}_2) \leq d_{SB}(\mathfrak{B}_1, \mathfrak{B}_2).$$

Proof. By Theorems 6.8 and 7.2, under the assumption $\mathfrak{n}(\mathfrak{B}_1) = \mathfrak{n}(\mathfrak{B}_2)$ we have $d_V(\mathfrak{B}_1, \mathfrak{B}_2) = d_V(\mathfrak{B}_1^*, \mathfrak{B}_2^*)$ and $d_V(\mathfrak{B}_1, \mathfrak{B}_1) \leq d_{SB}(\mathfrak{B}_1, \mathfrak{B}_1)$. Using (7.1) there exists $0 < c \leq 1$ such that $c d_Z(\mathfrak{B}_1^*, \mathfrak{B}_2^*) \leq d_V(\mathfrak{B}_1^*, \mathfrak{B}_2^*)$. This then implies $c d_{SB}(\mathfrak{B}_1, \mathfrak{B}_2) \leq d_V(\mathfrak{B}_1, \mathfrak{B}_2)$. \square

As a consequence, for any integer n the V-metric and the SB-metric considered as metrics on the subset $\mathcal{L}_{\text{cont}}^q(n)$ of all controllable behaviors of fixed McMillan degree n induce the same topology. Obviously this topology then also coincides with the one induced by the Z-metric on $\mathcal{L}_{\text{cont}}^q(n)$. The issue whether the Z-metric and the SB-metric define the same topology was posed as an open problem in [12, p. 1222]. Our result gives an answer to this question in full generality. The result was obtained before in [13] for input-output systems with stable transfer matrices.

8. Properties of the metrics. In this section we will take a closer look at the metrics introduced in sections 2 and 6 and establish several properties. Our main focus will be on expressing these properties in behavioral terms.

It is well known that for subspaces $\mathcal{V}_1, \mathcal{V}_2$ of the Euclidean space \mathbb{R}^n with the standard inner product we have

$$\text{gap}(\mathcal{V}_1, \mathcal{V}_2) < 1 \quad \text{if and only if} \quad \mathcal{V}_1 \cap \mathcal{V}_2^\perp = \{0\}.$$

In what follows we will study the question how this generalizes to the metrics that we defined on the space of controllable behaviors. In particular, for each of the metrics we have defined we will study the question, what are necessary and sufficient conditions under which the distance between two behaviors is strictly less than one?

8.1. Properties of the \mathcal{L}_2 -metric and the V-metric. We start off with answering the question posed in the introduction for the \mathcal{L}_2 -metric. The following lemma gives necessary and sufficient conditions in terms of the rational matrices appearing in image representations of the behaviors.

LEMMA 8.1. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$ with $\mathfrak{m}(\mathfrak{B}_1) = \mathfrak{m}(\mathfrak{B}_2)$. Let $G_1, G_2 \in R\mathcal{L}_\infty$ such that $\mathfrak{B}_1 = \text{im } G_1(\frac{d}{dt})$ and $\mathfrak{B}_2 = \text{im } G_2(\frac{d}{dt})$ with G_1, G_2 inner. Then the following three statements are equivalent:*

1. $d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) < 1$,
2. $\det(G_2^* G_1)(i\omega) \neq 0$ for all $\omega \in \mathbb{R}$ and $G_2^* G_1$ is biproper,
3. $G_2^* G_1$ is nonsingular and $(G_2^* G_1)^{-1} \in R\mathcal{L}_\infty$.

Proof. Let $\tilde{G}_2 \in R\mathcal{L}_\infty$ be co-inner and such that $\mathfrak{B}_2 = \ker \tilde{G}_2(\frac{d}{dt})$. By Theorem 5.1, $d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) = \|\tilde{G}_2 G_1\|_\infty$. From the proof of Theorem 5.1, recall that

$$\|\tilde{G}_2 G_1\|_\infty = \sup_{\omega \in \mathbb{R}} \sigma_{\max}(\tilde{G}_2 G_1)(i\omega) = 1 - \inf_{\omega \in \mathbb{R}} \sigma_{\min}(G_2^* G_1)(i\omega).$$

Obviously, $\inf_{\omega \in \mathbb{R}} \sigma_{\min}((G_2^* G_1)(i\omega)) > 0$ if and only if $(G_2^* G_1)(i\omega)$ is nonsingular for all $\omega \in \mathbb{R}$ and $\lim_{\omega \rightarrow \infty} \sigma_{\min}((G_2^* G_1)(i\omega)) > 0$; equivalently, statement 2 holds. Clearly, 2 and 3 are equivalent. \square

The property of biproperness in the above can be characterized equivalently in terms of the associated behaviors as follows.

LEMMA 8.2. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$, $\mathfrak{m}(\mathfrak{B}_1) = \mathfrak{m}(\mathfrak{B}_2)$. Let $G_1, G_2 \in R\mathcal{L}_\infty$ be right prime such that $\mathfrak{B}_1 = \text{im } G_1(\frac{d}{dt})$ and $\mathfrak{B}_2 = \text{im } G_2(\frac{d}{dt})$. Then $G_2^* G_1$ is biproper if and only if $\mathfrak{B}_1 \cap \mathfrak{B}_2^*$ is autonomous and $\dim(\mathfrak{B}_1 \cap \mathfrak{B}_2^*) = \mathfrak{n}(\mathfrak{B}_1) + \mathfrak{n}(\mathfrak{B}_2)$.*

Proof. (\Rightarrow) By Lemma 6.6, $G_2^* G_1$ biproper implies that $\mathfrak{B}_1 \cap \mathfrak{B}_2^*$ is autonomous. Let $G_1 = MN^{-1}$ be a right coprime factorization over $\mathbb{R}[\xi]$, and let $G_2^* = P^{-1}Q$ a left coprime factorization over $\mathbb{R}[\xi]$. Now, $G_2^* G_1 = P^{-1}QM N^{-1}$. By biproperness we have $\deg \det(QM) = \deg \det(P) + \deg \det(N)$. Also, as in the proof of Lemma 6.6, we have $\mathfrak{B}_1 \cap \mathfrak{B}_2^* = M(\frac{d}{dt}) \ker(QM)(\frac{d}{dt})$. This implies that $\dim(\mathfrak{B}_1 \cap \mathfrak{B}_2^*) = \deg \det(QM)$. On the other hand, by Lemma 6.4, $\deg \det(P) = \mathfrak{n}(\mathfrak{B}_2^*) = \mathfrak{n}(\mathfrak{B}_2)$ and $\deg \det(N) = \mathfrak{n}(\mathfrak{B}_1)$.

(\Leftarrow) The converse implication is proven by reversing the above argument. \square

This immediately yields the following behavioral characterization.

THEOREM 8.3. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$, $\mathfrak{m}(\mathfrak{B}_1) = \mathfrak{m}(\mathfrak{B}_2)$. Then the following are equivalent:*

1. $\delta_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) < 1$,
2. $\mathfrak{B}_1 \cap \mathfrak{B}_2^*$ is autonomous and has no imaginary frequencies and $\dim(\mathfrak{B}_1 \cap \mathfrak{B}_2^*) = \mathfrak{n}(\mathfrak{B}_1) + \mathfrak{n}(\mathfrak{B}_2)$.

Proof. This follows by combining Lemmas 6.6, 8.1, and 8.2. \square

Remark 8.4. Note that this theorem gives necessary and sufficient conditions under which the gap between the subspaces $\mathfrak{B}_1 \cap \mathcal{L}_2(\mathbb{R})$ and $\mathfrak{B}_2 \cap \mathcal{L}_2(\mathbb{R})$ of the Hilbert space $\mathcal{L}_2(\mathbb{R})$ is smaller than 1. Interestingly, these conditions involve specific system theoretic properties of the underlying behaviors, in particular autonomy of an intersection, absence of nonzero periodic signals, and an equality involving McMillan degrees. In this section we want to extend the above results to the other three metrics. For the V-metric this extension is straightforward.

LEMMA 8.5. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$ and $\mathfrak{m}(\mathfrak{B}_1) = \mathfrak{m}(\mathfrak{B}_2)$. Let $G_1, G_2 \in R\mathcal{H}_\infty$ be inner and right prime (over $R\mathcal{H}_\infty$) such that $\mathfrak{B}_1 = \text{im } G_1(\frac{d}{dt})$ and $\mathfrak{B}_2 = \text{im } G_2(\frac{d}{dt})$. Then the following are equivalent:*

1. $d_V(\mathfrak{B}_1, \mathfrak{B}_2) < 1$,
2. $d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) < 1$ and $\text{wno det}(G_2^* G_1) = 0$.

If any of these conditions hold, then $d_V(\mathfrak{B}_1, \mathfrak{B}_2) = \delta_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2)$.

Proof. Assume $d_V(\mathfrak{B}_1, \mathfrak{B}_2) < 1$. Then obviously $d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) < 1$. By Definition 6.1 we also have $\text{wno det}(G_2^* G_1) = 0$. Conversely, if $d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) < 1$, then by Lemma 8.1 $\det(G_2^* G_1)(i\omega) \neq 0$ for all $\omega \in \mathbb{R}$. Together with $\text{wno det}(G_2^* G_1) = 0$ this yields $d_V(\mathfrak{B}_1, \mathfrak{B}_2) = d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) < 1$. \square

This immediately yields the following behavioral characterization for the distance between two controllable behaviors in the V-metric to be smaller than 1.

THEOREM 8.6. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$, $\mathfrak{m}(\mathfrak{B}_1) = \mathfrak{m}(\mathfrak{B}_2)$. Then the following are equivalent:*

1. $d_V(\mathfrak{B}_1, \mathfrak{B}_2) < 1$,
2. $d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) < 1$ and $\dim(\mathfrak{B}_1 \cap \mathfrak{B}_2^*)_{\text{anti}} = \mathfrak{n}(\mathfrak{B}_2)$,
3. $d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) < 1$ and $\dim(\mathfrak{B}_1 \cap \mathfrak{B}_2^*)_{\text{stab}} = \mathfrak{n}(\mathfrak{B}_1)$.

Proof. The equivalence of 1 and 2 follows by combining Lemmas 6.6 and 8.5. To prove the equivalence of 2 and 3 note that by Theorem 8.3, $d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) < 1$ implies

$\dim(\mathfrak{B}_1 \cap \mathfrak{B}_2^*) = \mathfrak{n}(\mathfrak{B}_1) + \mathfrak{n}(\mathfrak{B}_2)$. Then obviously $\dim(\mathfrak{B}_1 \cap \mathfrak{B}_2^*)_{\text{stab}} = \mathfrak{n}(\mathfrak{B}_1)$ if and only if $\dim(\mathfrak{B}_1 \cap \mathfrak{B}_2^*)_{\text{stab}} = \mathfrak{n}(\mathfrak{B}_1)$. \square

8.2. Properties of the Z-metric and the SB-metric. We now study the question posed in the introduction to this section for the Z-metric and the SB-metric. In particular we will establish behavioral characterizations for the properties $d_Z(\mathfrak{B}_1, \mathfrak{B}_2) < 1$ and $d_{SB}(\mathfrak{B}_1, \mathfrak{B}_2) < 1$. For these two metrics this issue is more involved than for the \mathcal{L}_2 -metric and the V-metric. Our route will be to first derive general Hilbert space characterizations and next translate these into behavioral terms. We first recall the notion of Fredholm operator (see [10], [21]).

DEFINITION 8.7. Let \mathcal{H} be a Hilbert space and $F : \mathcal{H} \rightarrow \mathcal{H}$ a bounded linear operator. F is called a Fredholm operator if $\text{im } F$ is closed and $\dim(\ker F)$ and $\text{codim}(\text{im } F)$ are finite.

Now, for an arbitrary Hilbert space \mathcal{H} and closed subspaces \mathcal{V}_1 and \mathcal{V}_2 of \mathcal{H} , the following conditions under which the gap between \mathcal{V}_1 and \mathcal{V}_2 is smaller than 1 are well known (see [21], [14]).

PROPOSITION 8.8. Let \mathcal{H} be a Hilbert space and let \mathcal{V}_1 and \mathcal{V}_2 be closed subspaces. Let $\Pi_{\mathcal{V}_2}|_{\mathcal{V}_1}$ be the orthogonal projection onto \mathcal{V}_2 restricted to \mathcal{V}_1 . Then the following statements are equivalent:

1. $\text{gap}(\mathcal{V}_1, \mathcal{V}_2) < 1$,
2. $\Pi_{\mathcal{V}_2}|_{\mathcal{V}_1}$ is Fredholm, $\mathcal{V}_1 \cap \mathcal{V}_2^\perp = \{0\}$, and $\mathcal{V}_2 \cap \mathcal{V}_1^\perp = \{0\}$.

Note that obviously $\ker \Pi_{\mathcal{V}_2}|_{\mathcal{V}_1} = \mathcal{V}_1 \cap \mathcal{V}_2^\perp$. This general result is immediately applicable in our context with Hilbert space $\mathcal{L}_2(\mathbb{R}^+)$. Denote the gap in this Hilbert space by $\text{gap}_{\mathcal{L}_2(\mathbb{R}^+)}$. Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$. For convenience, use the shorthand notation $\mathfrak{B}_1^2 := \mathfrak{B}_1 \cap \mathcal{L}_2(\mathbb{R}^+)$ and $\mathfrak{B}_2^2 := \mathfrak{B}_2 \cap \mathcal{L}_2(\mathbb{R}^+)$. By Definition 2.2, $d_Z(\mathfrak{B}_1, \mathfrak{B}_2) = \text{gap}_{\mathcal{L}_2}(\mathfrak{B}_1^2, \mathfrak{B}_2^2) = \text{gap}_{\mathcal{L}_2(\mathbb{R}^+)}(\mathfrak{B}_1^2, \mathfrak{B}_2^2)$ and hence we immediately conclude the following.

PROPOSITION 8.9. Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$. Then $d_Z(\mathfrak{B}_1, \mathfrak{B}_2) < 1$ if and only if

1. $\Pi_{\mathfrak{B}_2^2}|_{\mathfrak{B}_1^2}$ is Fredholm,
2. $\mathfrak{B}_1^2 \cap (\mathfrak{B}_2^2)^\perp = \{0\}$ and $\mathfrak{B}_2^2 \cap (\mathfrak{B}_1^2)^\perp = \{0\}$.

Our aim in what follows is to reformulate conditions 1 and 2, obtaining more transparent, behavioral, system theoretic ones, in line with the conditions that we obtained for the \mathcal{L}_2 -metric and the V-metric in the previous subsection. We will now first deal with the Fredholm condition 1. Surprisingly, it turns out that this condition is equivalent to the condition that the distance in the \mathcal{L}_2 -metric is less than one.

THEOREM 8.10. Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$. Let $G_1, G_2 \in R\mathcal{H}_\infty$ be inner and right prime over $R\mathcal{H}_\infty$ such that $\mathfrak{B}_1 = \text{im } G_1(\frac{d}{dt})$ and $\mathfrak{B}_2 = \text{im } G_2(\frac{d}{dt})$. Then $\Pi_{\mathfrak{B}_2^2}|_{\mathfrak{B}_1^2}$ is Fredholm if and only if $T_{G_2^*G_1}$ is Fredholm. If, in addition, $\mathfrak{m}(\mathfrak{B}_1) = \mathfrak{m}(\mathfrak{B}_2)$, then the following are equivalent:

1. $T_{G_2^*G_1}$ is Fredholm,
2. $G_2^*G_1$ is nonsingular and $(G_2^*G_1)^{-1} \in R\mathcal{L}_\infty$,
3. $d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) < 1$.

Proof. Recall that $\mathfrak{B}_1^2 = \text{im } T_{G_1}$ and $\mathfrak{B}_2^2 = \text{im } T_{G_2}$. It was shown in [21] and [14] that $\Pi_{\mathfrak{B}_2^2}|_{\mathfrak{B}_1^2}$ is Fredholm if and only if $T_{G_2^*G_1}$ is Fredholm. The condition that $T_{G_2^*G_1}$ is Fredholm can be expressed equivalently as the invertibility condition 2 on the rational matrix $G_2^*G_1$. Indeed, by [21, p. 39] (see also [4]), for a given square matrix $G \in R\mathcal{L}_\infty$ the Toeplitz operator T_G is Fredholm if and only if $\det G(i\omega) \neq 0$ for all $\omega \in \mathbb{R}$ and G has a proper inverse; equivalently G is nonsingular and $G^{-1} \in R\mathcal{L}_\infty$. By Lemma 8.1 this is equivalent to the condition that the distance between \mathfrak{B}_1 and \mathfrak{B}_2 in the \mathcal{L}_2 -metric is less than 1. \square

In the next section we will make a detailed study of condition 2 in Proposition 8.9 by, in fact, explicitly computing the subspace intersections $\mathfrak{B}_1^2 \cap (\mathfrak{B}_2^2)^\perp$ and $\mathfrak{B}_2^2 \cap (\mathfrak{B}_1^2)^\perp$ in terms of driving variable state space representations of the behaviors \mathfrak{B}_1 and \mathfrak{B}_2 . This will then yield behavioral as well as state space characterizations of the property $d_Z(\mathfrak{B}_1, \mathfrak{B}_2) < 1$.

To conclude this subsection, we take a brief look at the SB-metric. By Theorem 5.6, $d_{SB}(\mathfrak{B}_1, \mathfrak{B}_2) = d_Z(\mathfrak{B}_1^*, \mathfrak{B}_2^*)$, so a characterization of the property $d_{SB}(\mathfrak{B}_1, \mathfrak{B}_2) < 1$ can be obtained by applying the results obtained so far to the dual behaviors \mathfrak{B}_1^* and \mathfrak{B}_2^* . Using the fact that the \mathcal{L}_2 -metric is invariant under dualization, in this way we obtain that $d_{SB}(\mathfrak{B}_1, \mathfrak{B}_2) < 1$ if and only if $d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) < 1$, and the intersection conditions appearing in condition 2 of Proposition 8.9 hold with \mathfrak{B}_1 and \mathfrak{B}_2 replaced by \mathfrak{B}_1^* and \mathfrak{B}_2^* .

9. State space representations of the subspace intersections. In this section, we will establish representations of the subspace intersections $\mathfrak{B}_1^2 \cap (\mathfrak{B}_2^2)^\perp$ and $\mathfrak{B}_2^2 \cap (\mathfrak{B}_1^2)^\perp$ in terms of driving variable state representations of the underlying behaviors \mathfrak{B}_1 and \mathfrak{B}_2 . Using the fact that $\mathfrak{B}_1^2 = \text{im } T_{G_1}$ and $\mathfrak{B}_2^2 = \text{im } T_{G_2}$, it will turn out that the intersection $\mathfrak{B}_1^2 \cap (\mathfrak{B}_2^2)^\perp$ can be expressed in terms of the kernel of a suitable Toeplitz operator with an invertible symbol. We will study such Toeplitz operators in subsection 9.1. In subsection 9.2 we will review some basic material on driving variable and output nulling representations of behaviors. Then, in subsections 9.3 and 9.4, we will give the desired representations of the subspace intersections.

A basic result that we will be using in this section is the following.

LEMMA 9.1. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$. Let $G_1, G_2 \in R\mathcal{H}_\infty$ be right prime over $R\mathcal{H}_\infty$ and such that $\mathfrak{B}_1 = \text{im } G_1(\frac{d}{dt})$ and $\mathfrak{B}_2 = \text{im } G_2(\frac{d}{dt})$. Then*

$$\mathfrak{B}_1^2 \cap (\mathfrak{B}_2^2)^\perp = T_{G_1} \ker T_{G_2^* G_1}.$$

Proof. Let $w \in \mathfrak{B}_1^2 \cap (\mathfrak{B}_2^2)^\perp$. Then $w = T_{G_1} v$. Since $(\mathfrak{B}_2^2)^\perp = (\text{im } T_{G_2})^\perp = \ker T_{G_2}^* = \ker T_{G_2^*}$, we also have $T_{G_2^*} w = 0$. By Halmos' theorem, since G_2^* is antistable and G_1 is stable, we have $T_{G_2^*} T_{G_1} = T_{G_2^* G_1}$. Hence $T_{G_2^* G_1} v = T_{G_2^*} w = 0$ so $w \in T_{G_1} \ker T_{G_2^* G_1}$. The converse inclusion is proven by reversing this argument. \square

Since a necessary condition for $d_Z(\mathfrak{B}_1, \mathfrak{B}_2) < 1$ is that $d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) < 1$, equivalently, $G_2^* G_1$ is nonsingular and $(G_2^* G_1)^{-1} \in R\mathcal{L}_\infty$, in this section we will assume that the symbol of the Toeplitz operator $T_{G_2^* G_1}$ is invertible in $R\mathcal{L}_\infty$.

9.1. Computing the kernel of Toeplitz operators with invertible symbol. In this subsection we will, for a given invertible real rational matrix, compute the kernel of the associated Toeplitz operator in terms of the constant real matrices obtained from a state space realization of the rational matrix.

Let $G \in R\mathcal{L}_\infty$ be nonsingular such that $G^{-1} \in R\mathcal{L}_\infty$. Consider the Toeplitz operator $T_G : \mathcal{L}_2(\mathbb{R}^+) \rightarrow \mathcal{L}_2(\mathbb{R}^+)$ with symbol G . Let $G(s) = C(sI - A)^{-1}B + D$ be a realization, possibly nonminimal, where $A \in \mathbb{R}^{n \times n}$ has no imaginary axis eigenvalues. Since G is biproper, D is nonsingular. We denote by $\mathcal{X}_-(A)$ the stable subspace of A , i.e.,

$$\mathcal{X}_-(A) := \left\{ x_0 \in \mathbb{R}^n \mid \lim_{t \rightarrow \infty} e^{At} x_0 = 0 \right\}.$$

Likewise, $\mathcal{X}_+(A)$ denotes the antistable subspace of A . Clearly $\mathbb{R}^n = \mathcal{X}_-(A) \oplus \mathcal{X}_+(A)$. Let $\mathbb{1}(t)$ be the indicator function of \mathbb{R}^+ . Denote the unobservable subspace of the

pair $(-D^{-1}C, A - BD^{-1}C)$ by \mathcal{N} . We now compute the kernel $\ker T_G$ of the Toeplitz operator T_G .

THEOREM 9.2. *Let $\mathcal{X}_0 := [\mathcal{X}_-(A - BD^{-1}C) + \mathcal{N}] \cap \mathcal{X}_+(A)$. Then the kernel of T_G is equal to*

$$\ker T_G = \{v \in \mathcal{L}_2(\mathbb{R}^+) \mid \exists x_0 \in \mathcal{X}_0 : v(t) = -\mathbb{1}(t)D^{-1}Ce^{(A-BD^{-1}C)t}x_0\}.$$

Consequently $\ker(T_G)$ is a finite dimensional subspace of $\mathcal{L}_2(\mathbb{R}^+)$ with dimension equal to $\dim(\mathcal{X}_0) - \dim(\mathcal{X}_+(A) \cap \mathcal{N})$.

Proof. Let $v \in \mathcal{L}_2(\mathbb{R}^+)$ be in $\ker T_G$. Let $w = M_G v$. This w is the unique $w \in \mathcal{L}_2(\mathbb{R})$ given by $\dot{x} = Ax + Bv$, $w = Cx + Dv$. Since $\Pi_+ w = 0$, we must have that $w(t) = 0$ for $t \geq 0$. Using the fact that D is nonsingular, this implies that for $t \geq 0$ we have $v(t) = -D^{-1}Ce^{(A-BD^{-1}C)t}x(0)$. Since $v \in \mathcal{L}_2(\mathbb{R}^+)$ we must have $v(t) \rightarrow 0$ as $t \rightarrow \infty$. This implies that $x(0)$ must be contained in $\mathcal{X}_-(A - BD^{-1}C) + \mathcal{N}$, the sum of the stable subspace and the unobservable subspace. Next we prove that $x(0) \in \mathcal{X}_+(A)$. Let S be a coordinate transformation in \mathbb{R}^n such that

$$S^{-1}AS = \begin{pmatrix} A_- & 0 \\ 0 & A_+ \end{pmatrix}, \quad S^{-1}B = \begin{pmatrix} B_- \\ B_+ \end{pmatrix},$$

with A_- Hurwitz and A_+ anti-Hurwitz. Partition $S = (S_- \ S_+)$. Then $\text{im}(S_+)$ is equal to the $\mathcal{X}_+(A)$. For $t \geq 0$ the state trajectory $x(t)$ is explicitly given by

$$x(t) = \int_0^t S_- e^{A_-(t-s)} B_- v(s) ds + \int_t^\infty S_+ e^{A_+(t-s)} B_+ v(s) ds,$$

where the integration starts at 0 due to the fact that $v(t) = 0$ for $t \leq 0$. By evaluating $x(t)$ at $t = 0$ this yields

$$x(0) = \int_0^\infty S_+ e^{A_+(t-s)} B_+ v(s) ds,$$

which obviously is contained in $\text{im}(S_+) = \mathcal{X}_+(A)$.

Conversely, let $v(t) = -\mathbb{1}(t)D^{-1}Ce^{(A-BD^{-1}C)t}x_0$ with $x_0 \in \mathcal{X}_0$. Let $w = M_G v$. Again, this w is the unique $w \in \mathcal{L}_2(\mathbb{R})$ given by $\dot{x} = Ax + Bv$, $w = Cx + Dv$. We claim that, in fact, w is given by

$$w(t) = \begin{cases} Ce^{At}x_0, & t < 0, \\ 0, & t \geq 0. \end{cases}$$

First note that since $x_0 \in \mathcal{X}_+(A)$, this w is in $\mathcal{L}_2(\mathbb{R})$. We will now to prove that w satisfies the equations $\dot{x} = Ax + Bv$, $w = Cx + Dv$ with $x(0) = x_0$. Indeed, for $t < 0$ it is given that $v(t) = 0$. Thus the equations become $\dot{x}(t) = Ax(t)$, $w(t) = Cx(t)$, which are indeed satisfied for $t < 0$ by the given w . For $t \geq 0$, define $x(t) := e^{(A-BD^{-1}C)t}x_0$. Then $v(t) = -D^{-1}Cx(t)$ and hence $\dot{x}(t) = Ax(t) + Bv(t)$ for $t \geq 0$. Finally, $0 = Cx(t) + Dv(t)$ for $t \geq 0$.

We have now shown that $w(t) = 0$ for $t \geq 0$ so $\Pi_+ w = 0$. This implies $\Pi_+ M_G v = 0$, so $v \in \ker(T_G)$.

Let $\{x_i, i = 1, 2, \dots, r\}$ be a basis for the subspace $\mathcal{X}_0 \cap \mathcal{N}$ and extend it to a basis $\{x_i, i = 1, 2, \dots, k\}$ of \mathcal{X}_0 . Then a basis for $\ker(T_G)$ is given by

$$\{-\mathbb{1}(t)D^{-1}Ce^{(A-BD^{-1}C)t}x_i, i = r + 1, 2, \dots, k\}.$$

Thus $\dim(\ker(T_G)) = \dim(\mathcal{X}_0) - \dim(\mathcal{X}_0 \cap \mathcal{N})$. The result then follows from the observation that $\mathcal{X}_0 \cap \mathcal{N} = \mathcal{X}_+(A) \cap \mathcal{N}$. \square

9.2. Driving variable and output nulling representations of behaviors.

We will now review some basic facts on driving variable and output nulling representations of behaviors. For details we refer to [25], [20], [16]. We first consider driving variable representations.

Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{q \times n}$, $D \in \mathbb{R}^{q \times m}$, and consider the equations

$$(9.1) \quad \dot{x} = Ax + Bv, \quad w = Cx + Dv.$$

These equations represent the so-called full behavior

$$(9.2) \quad \mathfrak{B}_{DV}(A, B, C, D) := \{(w, x, v) \in \mathcal{L}_{\text{loc}}(\mathbb{R}, \mathbb{R}^q) \times \mathcal{L}_{\text{loc}}(\mathbb{R}, \mathbb{R}^n) \times \mathcal{L}_{\text{loc}}(\mathbb{R}, \mathbb{R}^m) \mid (9.1) \text{ holds}\}.$$

In (9.1), we interpret w as manifest variable and (x, v) as latent variables. Thus, \mathfrak{B}_{DV} is a latent variable representation of its external behavior given by

$$(9.3) \quad \mathfrak{B}_{DV}(A, B, C, D)_{\text{ext}} = \{w \mid \exists (x, v) \text{ such that } (w, x, v) \in \mathfrak{B}_{DV}(A, B, C, D)\}.$$

In fact, in (9.1), x is a state variable and v is an auxiliary variable, called the *driving variable*. Further, if $\mathfrak{B} = \mathfrak{B}_{DV}(A, B, C, D)_{\text{ext}}$, then we call \mathfrak{B}_{DV} a *driving variable representation* of \mathfrak{B} . A driving variable representation of \mathfrak{B} is called *minimal* if the state dimension n and the driving variable dimension m are minimal over all driving variable representations. It can be shown that this holds if and only if $n = \mathbf{n}(\mathfrak{B})$ and $m = \mathbf{m}(\mathfrak{B})$.

Next we review output nulling representations. Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times q}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times q}$ and consider the equations

$$(9.4) \quad \dot{x} = Ax + Bw, \quad 0 = Cx + Dw.$$

The full behavior represented by these equations is given by

$$(9.5) \quad \mathfrak{B}_{ON}(A, B, C, D) := \{(w, x) \in \mathcal{L}_{\text{loc}}(\mathbb{R}, \mathbb{R}^q) \times \mathcal{L}_{\text{loc}}^1(\mathbb{R}, \mathbb{R}^n) \mid (9.4) \text{ holds}\}.$$

In (9.4), we interpret w as manifest variable and x as a latent variable. Thus, \mathfrak{B}_{ON} is a latent variable representation of its external behavior given by

$$(9.6) \quad \mathfrak{B}_{ON}(A, B, C, D)_{\text{ext}} = \{w \mid \exists x \text{ such that } (w, x) \in \mathfrak{B}_{ON}(A, B, C, D)\}.$$

Also in (9.4), x is a state variable. If $\mathfrak{B} = \mathfrak{B}_{ON}(A, B, C, D)_{\text{ext}}$, then we call \mathfrak{B}_{ON} an *output nulling representation* of \mathfrak{B} . An output nulling representation of \mathfrak{B} is called *minimal* if the state dimension n and the dimension of the equation space p are minimal over all output nulling representations. This holds if and only if $n = \mathbf{n}(\mathfrak{B})$ and $p = q - \mathbf{m}(\mathfrak{B})$.

There is the following duality between driving variable and output nulling representations: if $\mathfrak{B} \in \mathcal{L}_{\text{cont}}^q$, then $\mathfrak{B}_{DV}(A, B, C, D)$ is a minimal DV representation of \mathfrak{B} if and only if $\mathfrak{B}_{ON}(-A^\top, C^\top, -B^\top, D^\top)$ is a minimal output nulling representation of the dual behavior \mathfrak{B}^* .

The following result from [16, Theorem 5.8] will be used in the rest of the paper.

LEMMA 9.3. *Let $G \in \mathbb{R}(\xi)_P$. Let $G(\xi) = C(\xi I - A)^{-1}B + D$ be a realization with (A, B) a controllable pair and (C, A) an observable pair. Then $\mathfrak{B}_{DV}(A, B, C, D)$ is a minimal DV-representation of $\text{im } G(\frac{d}{dt})$ if and only if G is right prime over $\mathbb{R}(\xi)_P$ and has no zeros.*

9.3. State representation of $\mathfrak{B}_1 \cap \mathfrak{B}_2^*$ and $\mathfrak{B}_2 \cap \mathfrak{B}_1^*$. In this subsection we will establish state representations of $\mathfrak{B}_1 \cap \mathfrak{B}_2^*$ and $\mathfrak{B}_2 \cap \mathfrak{B}_1^*$ in terms of realizations of rational image representations of \mathfrak{B}_1 and \mathfrak{B}_2 .

Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$ with $\mathfrak{m}(\mathfrak{B}_1) = \mathfrak{m}(\mathfrak{B}_2)$. A standing assumption throughout this section will be that $G_1, G_2 \in R\mathcal{H}_\infty$ are right prime over $R\mathcal{H}_\infty$, $\mathfrak{B}_1 = \text{im } G_1(\frac{d}{dt})$ and $\mathfrak{B}_2 = \text{im } G_2(\frac{d}{dt})$, and G_1 and G_2 have no zeros. According to Theorem 3.2, such G_1 and G_2 exist. Now realize $G_i(\xi) = C_i(\xi I - A_i)^{-1}B_i + D_i$, $i = 1, 2$, with (A_i, B_i) controllable and (C_i, A_i) observable. We then have $G_2^*(\xi) = -B_2^\top(\xi I + A_2^\top)^{-1}C_2^\top + D_2^\top$. According to Lemma 9.3, this yields the following minimal driving variable representation of \mathfrak{B}_1 :

$$(9.7) \quad \dot{x}_1 = A_1 x_1 + B_1 v, \quad w_1 = C_1 x_1 + D_1 v.$$

Furthermore, a minimal output nulling representation of \mathfrak{B}_2^* is given by

$$(9.8) \quad \dot{x}_2 = -A_2^\top x_1 + C_2^\top w_2, \quad 0 = -B_2^\top x_2 + D_2^\top w_2.$$

Now define

$$(9.9) \quad A := \begin{pmatrix} A_1 & 0 \\ C_2^\top C_1 & -A_2^\top \end{pmatrix}, \quad B := \begin{pmatrix} B_1 \\ C_2^\top D_1 \end{pmatrix}, \quad C := (D_2^\top C_1 \quad -B_2^\top), \quad D := D_2^\top D_1.$$

Clearly, then $(G_2^* G_1)(\xi) = C(\xi I - A)^{-1}B + D$. Also, it is easily verified that a state representation of the intersection $\mathfrak{B}_1 \cap \mathfrak{B}_2^*$ is given by

$$(9.10) \quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ C_2^\top C_1 & -A_2^\top \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ C_2^\top D_1 \end{pmatrix} v,$$

$$(9.11) \quad 0 = (D_2^\top C_1 \quad -B_2^\top) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + D_2^\top D_1 v,$$

$$(9.12) \quad w = C_1 x_1 + D_1 v,$$

i.e., $w \in \mathfrak{B}_1 \cap \mathfrak{B}_2^*$ if and only if there exists x_1, x_2 and v such that (9.10), (9.11), and (9.12) hold.

If we assume that $(G_2^* G_1)^{-1} \in R\mathcal{L}_\infty$, then $D = D_2^\top D_1$ is nonsingular. Eliminating the variable v from (9.10), (9.11), and (9.12) and writing $x = \text{col}(x_1, x_2)$, an alternative state representation of $\mathfrak{B}_1 \cap \mathfrak{B}_2^*$ is then given by

$$(9.13) \quad \dot{x} = (A - BD^{-1}C)x, \quad w = ((C_1 \ 0) - D_1 D^{-1}C)x.$$

We now address the issue of minimality of the state representation (9.13). Let A_i be an $n_i \times n_i$ matrix ($i = 1, 2$). By minimality of the driving variable and output nulling representations (9.7) and (9.8) above, we have $\mathfrak{n}(\mathfrak{B}_1) = n_1$ and $\mathfrak{n}(\mathfrak{B}_2^*) = n_2$. Thus we obtain the next lemma.

LEMMA 9.4. Assume that $(G_2^*G_1)^{-1} \in R\mathcal{L}_\infty$. Then (9.13) is a minimal state representation of $\mathfrak{B}_1 \cap \mathfrak{B}_2^*$.

Proof. By Lemma 8.2, $\mathfrak{B}_1 \cap \mathfrak{B}_2^*$ is autonomous, and we have $\dim(\mathfrak{B}_1 \cap \mathfrak{B}_2^*) = \mathbf{n}(\mathfrak{B}_1) + \mathbf{n}(\mathfrak{B}_2)$. As a consequence, since $\mathbf{n}(\mathfrak{B}_2^*) = \mathbf{n}(\mathfrak{B}_2)$, the state space dimension of the state representation (9.13), being equal to $n_1 + n_2$, is equal to the McMillan degree $\dim(\mathfrak{B}_1 \cap \mathfrak{B}_2^*)$ of $\mathfrak{B}_1 \cap \mathfrak{B}_2^*$, and hence the state representation is minimal. \square

In particular this implies that the state is observable from the manifest variable, so for any w there exists exactly one $x = \text{col}(x_1, x_2)$ such that (9.13) holds. Observability is equivalent to observability of the pair $((C_1 \ 0) - D_1 D^{-1}C, A - BD^{-1}C)$.

Clearly, using observability, the stable part $(\mathfrak{B}_1 \cap \mathfrak{B}_2^*)_{\text{stab}}$ of the autonomous behavior $\mathfrak{B}_1 \cap \mathfrak{B}_2^*$ consists of those external trajectories w whose corresponding state trajectory x passes through the stable subspace of $A - BD^{-1}C$; in other words,

$$(9.14) \quad (\mathfrak{B}_1 \cap \mathfrak{B}_2^*)_{\text{stab}} = \{w \in \mathfrak{B}_1 \cap \mathfrak{B}_2^* \mid x(0) \in \mathcal{X}_-(A - BD^{-1}C)\}.$$

Of course, a similar representation holds for the antistable part $(\mathfrak{B}_1 \cap \mathfrak{B}_2^*)_{\text{anti}}$.

The following lemma states that although $(-D^{-1}C, A - BD^{-1}C)$ does not need to be observable, we do have that it is *detectable*.

LEMMA 9.5. Assume that $(G_2^*G_1)^{-1} \in R\mathcal{L}_\infty$. Let \mathcal{N} be the unobservable subspace of the pair $(-D^{-1}C, A - BD^{-1}C)$. Then $\mathcal{N} \subset \mathcal{X}_-(A - BD^{-1}C)$.

Proof. Under the assumption, the state representation (9.13) is minimal. Let $x_0 \in \mathcal{N}$, $x_0 = \text{col}(x_{10}, x_{20})$, and let w be the corresponding external trajectory. In (9.13) we then have $-D^{-1}Cx = 0$, so $w = C_1x_1$, with $\dot{x}_1 = A_1x_1$. Since A_1 is Hurwitz, $w(t) \rightarrow 0$ as $t \rightarrow \infty$. By observability of (9.13) this implies $x_0 \in \mathcal{X}_-(A - BD^{-1}C)$. \square

Note that x_0 is of the form $\text{col}(0, x_{20})$ if and only if $x_0 \in \mathcal{X}_+(A)$. The following will be very useful.

LEMMA 9.6. Assume that $(G_2^*G_1)^{-1} \in R\mathcal{L}_\infty$. Then $\mathcal{X}_+(A) \cap \mathcal{N} = \{0\}$.

Proof. Let $x_0 \in \mathcal{X}_+(A)$ and in (9.13) assume that the corresponding state trajectory x satisfies $D^{-1}Cx = 0$. Since then $\dot{x}_1 = A_1x_1$ and since $x_0 = (0, x_{20})$ we find that $x_1 = 0$, so x_2 satisfies $\dot{x}_2 = -A_2^\top x_2, 0 = -B_2^\top x_2$. As (A_2, B_2) was chosen to be controllable, this implies that $x_2 = 0$, which yields $x_0 = 0$. \square

As a consequence of the representation (9.14) we can also compute the following.

LEMMA 9.7. $\dim(\mathfrak{B}_1 \cap \mathfrak{B}_2^*)_{\text{stab}} = \dim \mathcal{X}_-(A - BD^{-1}C)$.

We now turn to establishing a state representation of $\mathfrak{B}_2 \cap \mathfrak{B}_1^*$. As before, let $G_i(\xi) = C_i(\xi I - A_i)^{-1}B_i + D_i$, $i = 1, 2$ with (A_i, B_i) controllable and (C_i, A_i) observable. Now define

$$(9.15) \quad A' := \begin{pmatrix} A_2 & 0 \\ C_1^\top C_2 & -A_1^\top \end{pmatrix}, B' := \begin{pmatrix} B_2 \\ C_1^\top D_2 \end{pmatrix}, C' := (D_1^\top C_2 \quad -B_1^\top), D' := D_1^\top D_2.$$

Under the assumption $(G_1^*G_2)^{-1} \in R\mathcal{L}_\infty$ (equivalently $(G_2^*G_1)^{-1} \in R\mathcal{L}_\infty$), a minimal state representation of the autonomous behavior $\mathfrak{B}_2 \cap \mathfrak{B}_1^*$ is then given by

$$(9.16) \quad \frac{d}{dt}x' = (A' - B'D'^{-1}C')x', \quad w' = ((C_2 \ 0) - D_2D'^{-1}C')x',$$

where $x' = \text{col}(x_2, x_1)$. Our aim is to relate this explicitly to the state representation (9.13) of $\mathfrak{B}_1 \cap \mathfrak{B}_2^*$. Indeed, the quadruple (9.15) is similar to the dual of (A, B, C, D) : if we define

$$S := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix},$$

then $SA'S^{-1} = -A^\top$, $SB' = -C^\top$, $C'S^{-1} = B^\top$, and $D' = D^\top$. Thus, introducing the new state variable $z := Sx'$, we get the following minimal state representation for $\mathfrak{B}_2 \cap \mathfrak{B}_1^*$:

$$(9.17) \quad \dot{z} = -(A - BD^{-1}C)^\top z, \quad w' = ((C_2 \ 0) - D_2 D^{-\top} B^\top) z.$$

The advantage of this representation is that it explicitly displays its relation with the state representation (9.13) of $\mathfrak{B}_1 \cap \mathfrak{B}_2^*$. This will be useful in what follows. Note that this representation is again observable. Therefore

$$(9.18) \quad (\mathfrak{B}_2 \cap \mathfrak{B}_1^*)_{\text{stab}} = \{w' \in \mathfrak{B}_2 \cap \mathfrak{B}_1^* \mid z(0) \in \mathcal{X}_-(-(A - BD^{-1}C)^\top)\}.$$

Since $\mathcal{X}_-(-(A - BD^{-1}C)^\top) = \mathcal{X}_-(A - BD^{-1}C)^\perp$, in addition to the result of Lemma 9.7 we have

$$(9.19) \quad \dim(\mathfrak{B}_2 \cap \mathfrak{B}_1^*)_{\text{stab}} = n_1 + n_2 - \dim \mathcal{X}_-(A - BD^{-1}C).$$

9.4. Representation of $\mathfrak{B}_1^2 \cap (\mathfrak{B}_2^2)^\perp$ and $\mathfrak{B}_2^2 \cap (\mathfrak{B}_1^2)^\perp$. In this final subsection we wrap up things and establish explicit representations of the subspace intersections $\mathfrak{B}_1^2 \cap (\mathfrak{B}_2^2)^\perp$ and $\mathfrak{B}_2^2 \cap (\mathfrak{B}_1^2)^\perp$ in terms of realizations of rational image representations of \mathfrak{B}_1 and \mathfrak{B}_2 .

Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$, $\mathfrak{m}(\mathfrak{B}_1) = \mathfrak{m}(\mathfrak{B}_2)$. Recall the shorthand notation $\mathfrak{B}_1^2 := \mathfrak{B}_1 \cap \mathcal{L}_2(\mathbb{R}^+)$ and $\mathfrak{B}_2^2 := \mathfrak{B}_2 \cap \mathcal{L}_2(\mathbb{R}^+)$. Again, let $G_1, G_2 \in R\mathcal{H}_\infty$ be right prime over $R\mathcal{H}_\infty$ such that $\mathfrak{B}_1 = \text{im } G_1(\frac{d}{dt})$ and $\mathfrak{B}_2 = \text{im } G_2(\frac{d}{dt})$ and such that G_1 and G_2 have no zeros. Recall from Lemma 9.1 that $\mathfrak{B}_1^2 \cap (\mathfrak{B}_2^2)^\perp = T_{G_1} \ker T_{G_2^* G_1}$. Under the additional condition that $(G_2^* G_1)^{-1} \in R\mathcal{L}_\infty$ (equivalently, $d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) < 1$; see Theorem 8.10), a minimal state representation of $\mathfrak{B}_1 \cap \mathfrak{B}_2^*$ is given by (9.13).

In finding a representation of the intersection $\mathfrak{B}_1^2 \cap (\mathfrak{B}_2^2)^\perp$, the subbehavior of $\mathfrak{B}_1 \cap \mathfrak{B}_2^*$ of all external trajectories w whose corresponding state trajectory x passes through both the intersection of the stable subspace of $A - BD^{-1}C$ and the anti-stable subspace of A turns out to be crucial. Define

$$(9.20) \quad (\mathfrak{B}_1 \cap \mathfrak{B}_2^*)^+ := \{w \in \mathfrak{B}_1 \cap \mathfrak{B}_2^* \mid x(0) \in \mathcal{X}_+(A)\}.$$

Note that $\dim(\mathfrak{B}_1 \cap \mathfrak{B}_2^*)^+ = n_2$, the MacMillan degree of \mathfrak{B}_2 .

In what follows, let $P_+ : (\mathbb{R}^q)^\mathbb{R} \rightarrow (\mathbb{R}^q)^\mathbb{R}$ denote the map that projects functions from \mathbb{R} to \mathbb{R}^q onto their future: $(P_+ w)(t) := w(t)\mathbb{1}(t)$. Then, by applying Theorem 9.2 we find that the subspace $\mathfrak{B}_1^2 \cap (\mathfrak{B}_2^2)^\perp$ is the image of $(\mathfrak{B}_1 \cap \mathfrak{B}_2^*)_{\text{stab}} \cap (\mathfrak{B}_1 \cap \mathfrak{B}_2^*)^+$ under this projection.

THEOREM 9.8. Assume $(G_2^* G_1)^{-1} \in R\mathcal{L}_\infty$. Let A, B, C , and D be given by (9.9). Then we have

$$\mathfrak{B}_1^2 \cap (\mathfrak{B}_2^2)^\perp = P_+ ((\mathfrak{B}_1 \cap \mathfrak{B}_2^*)_{\text{stab}} \cap (\mathfrak{B}_1 \cap \mathfrak{B}_2^*)^+).$$

Furthermore, the dimension of $\mathfrak{B}_1^2 \cap (\mathfrak{B}_2^2)^\perp$ is equal to $\dim \mathcal{X}_-(A - BD^{-1}C) \cap \mathcal{X}_+(A)$.

Proof. If $w_1 \in \mathfrak{B}_1^2 \cap (\mathfrak{B}_2^2)^\perp$, then it is of the form $w_1 = T_{G_1} v_1$ with $v_1 \in \ker T_{G_2^* G_1}$. By Theorem 9.2 and Lemma 9.5 we have $v_1(t) = -\mathbb{1}(t)v(t)$ with

$$v(t) = D^{-1}C e^{(A - BD^{-1}C)t} x_0$$

for some $x_0 \in \mathcal{X}_-(A - BD^{-1}C) \cap \mathcal{X}_+(A)$. Since $G_1 \in R\mathcal{H}_\infty$ we have $w_1 = T_{G_1} v_1 = M_{G_1} v_1$, which is the unique solution in $\mathcal{L}_2(\mathbb{R})$ of $\dot{x}_1 = A_1 x_1 + B_1 v_1$, $w_1 = C_1 x_1 + D_1 v_1$.

For $t \geq 0$ we have $v_1(t) = v(t)$, so for $t \geq 0$ we must have $w_1(t) = w(t)$, where $w(t)$ is determined by the equations $\dot{x} = (A - BD^{-1}C)x$, $x(0) = x_0$, $w = ((C_1 \ 0) - D_1 D^{-1}C)x$. Clearly, $w \in (\mathfrak{B}_1 \cap \mathfrak{B}_2^*)_{\text{stab}} \cap (\mathfrak{B}_1 \cap \mathfrak{B}_2^*)^+$ and $w_1 = P_+ w$.

Conversely, let $w \in (\mathfrak{B}_1 \cap \mathfrak{B}_2^*)_{\text{stab}} \cap (\mathfrak{B}_1 \cap \mathfrak{B}_2^*)^+$ and consider $P_+ w$. By definition, w is determined by the equations $\dot{x} = (A - BD^{-1}C)x$, $x(0) = x_0$, $w = ((C_1 \ 0) - D_1 D^{-1}C)x$ with $x_0 \in \mathcal{X}_-(A - BD^{-1}C) \cap \mathcal{X}_+(A)$. Define

$$v(t) := -D^{-1}Cx(t).$$

Then by Theorem 9.2 and Lemma 9.5 we have $v_1(t) = -\mathbb{1}(t)v(t) \in \ker T_{G_2^* G_1}$. Define $w_1 := T_{G_1} v_1$. Then, again, $w_1 = M_{G_1} v_1$ is the unique trajectory in $\mathcal{L}_2(\mathbb{R})$ that satisfies the equations $\dot{x}_1 = A_1 x_1 + B_1 v_1$, $w_1 = C_1 x_1 + D_1 v_1$. Since for $t \geq 0$ we have $v_1(t) = v(t)$, we must also have $w_1(t) = w(t)$ for $t \geq 0$, so $P_+ w = w_1 \in T_{G_1} \ker T_{G_2^* G_1} = w_1 \in \mathfrak{B}_1^2 \cap (\mathfrak{B}_2^2)^\perp$.

Finally, from observability of the state representation (9.13) the dimension of $(\mathfrak{B}_1 \cap \mathfrak{B}_2^*)_{\text{stab}} \cap (\mathfrak{B}_1 \cap \mathfrak{B}_2^*)^+$ is equal to $\dim(\mathcal{X}_-(A - BD^{-1}C) \cap \mathcal{X}_+(A))$. Obviously this must also be the dimension of $\mathfrak{B}_1^2 \cap (\mathfrak{B}_2^2)^\perp$. \square

Next we turn to representing the dual intersection $\mathfrak{B}_2^2 \cap (\mathfrak{B}_1^2)^\perp$. Thus, as before, we introduce the subbehavior of $\mathfrak{B}_2 \cap \mathfrak{B}_1^*$ of all external trajectories w' such that their corresponding state trajectory x' passes through $\mathcal{X}_+(A')$ (with respect to the state representation (9.16)):

$$(9.21) \quad (\mathfrak{B}_2 \cap \mathfrak{B}_1^*)^+ := \{w' \in \mathfrak{B}_2 \cap \mathfrak{B}_1^* \mid x'(0) \in \mathcal{X}_+(A')\}.$$

It is easily verified that in terms of the alternative state representation (9.17) we have

$$(9.22) \quad (\mathfrak{B}_2 \cap \mathfrak{B}_1^*)^+ = \{w' \in \mathfrak{B}_2 \cap \mathfrak{B}_1^* \mid z(0) \in \mathcal{X}_+(-A^\top)\},$$

so analogously to Theorem 9.8 we find that if $(G_2^* G_1)^{-1} \in R\mathcal{L}_\infty$, then

$$\mathfrak{B}_2^2 \cap (\mathfrak{B}_1^2)^\perp = P_+ ((\mathfrak{B}_2 \cap \mathfrak{B}_1^*)_{\text{stab}} \cap (\mathfrak{B}_2 \cap \mathfrak{B}_1^*)^+).$$

Moreover, the dimension of $\mathfrak{B}_2^2 \cap (\mathfrak{B}_1^2)^\perp$ is equal to $\dim \mathcal{X}_+(-(A - BD^{-1}C)^\top) \cap \mathcal{X}_+(-A^\top)$.

10. Properties of the metrics, continued. Using the detailed analysis in the previous section, we will now continue our study of the Z-metric. We will also return to the \mathcal{L}_2 -metric, the V-metric, and the SB-metric. Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$ with $\mathfrak{m}(\mathfrak{B}_1) = \mathfrak{m}(\mathfrak{B}_2)$. Again, a standing assumption throughout this section will be that $G_1, G_2 \in R\mathcal{H}_\infty$ are right prime over $R\mathcal{H}_\infty$, $\mathfrak{B}_1 = \text{im } G_1(\frac{d}{dt})$ and $\mathfrak{B}_2 = \text{im } G_2(\frac{d}{dt})$, and G_1 and G_2 have no zeros. In addition we now assume that both G_1 and G_2 are inner. Throughout this section, the constant real matrices A, B, C , and D are obtained from minimal realizations of G_1 and G_2 and are given by (9.9).

10.1. Behavioral characterizations for Z-metric and V-metric. From Theorem 8.6 recall that for controllable behaviors $\mathfrak{B}_1, \mathfrak{B}_2$ with the same number of inputs, $d_V(\mathfrak{B}_1, \mathfrak{B}_2) < 1$ if and only if $d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) < 1$ and the dimension of $(\mathfrak{B}_1 \cap \mathfrak{B}_2^*)_{\text{stab}}$ is equal to the MacMillan degree of \mathfrak{B}_2 . Also for the Z-metric, conditions can now be formulated in terms of the stable part of $\mathfrak{B}_1 \cap \mathfrak{B}_2^*$. First note the following immediate consequence of Theorem 9.8.

LEMMA 10.1. *Assume that $(G_2^* G_1)^{-1} \in R\mathcal{L}_\infty$. Let $(\mathfrak{B}_1 \cap \mathfrak{B}_2^*)^+$ be defined by (9.20). Then $\mathfrak{B}_1^2 \cap (\mathfrak{B}_2^2)^\perp = 0$ if and only if $(\mathfrak{B}_1 \cap \mathfrak{B}_2^*)_{\text{stab}} \cap (\mathfrak{B}_1 \cap \mathfrak{B}_2^*)^+ = \{0\}$.*

By applying Proposition 8.9 and Theorem 9.8, this immediately leads to the following behavioral characterization. Again, let $(\mathfrak{B}_1 \cap \mathfrak{B}_2^*)^+$ be defined by (9.20) and let $(\mathfrak{B}_2 \cap \mathfrak{B}_1^*)^+$ be defined by (9.21).

THEOREM 10.2. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$ with $\mathfrak{m}(\mathfrak{B}_1) = \mathfrak{m}(\mathfrak{B}_2)$. Then $d_Z(\mathfrak{B}_1, \mathfrak{B}_2) < 1$ if and only if the following three conditions hold:*

1. $d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) < 1$,
2. $(\mathfrak{B}_1 \cap \mathfrak{B}_2^*)_{\text{stab}} \cap (\mathfrak{B}_1 \cap \mathfrak{B}_2^*)^+ = \{0\}$,
3. $(\mathfrak{B}_2 \cap \mathfrak{B}_1^*)_{\text{stab}} \cap (\mathfrak{B}_2 \cap \mathfrak{B}_1^*)^+ = \{0\}$.

Also in case of the V-metric, the behavior intersections appearing in conditions 2 and 3 of Theorem 10.2 turn out to be crucial in order to characterize distance less than one. Indeed, the following theorem gives an alternative for the characterization of Theorem 8.6 (see also [21, p. 41]).

THEOREM 10.3. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$ with $\mathfrak{m}(\mathfrak{B}_1) = \mathfrak{m}(\mathfrak{B}_2)$. Then $d_V(\mathfrak{B}_1, \mathfrak{B}_2) < 1$ if and only if the following two conditions hold:*

1. $d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) < 1$,
2. $\dim(\mathfrak{B}_1 \cap \mathfrak{B}_2^*)_{\text{stab}} \cap (\mathfrak{B}_1 \cap \mathfrak{B}_2^*)^+ = \dim(\mathfrak{B}_2 \cap \mathfrak{B}_1^*)_{\text{stab}} \cap (\mathfrak{B}_2 \cap \mathfrak{B}_1^*)^+$.

Condition 2 above is equivalent to

$$\dim \mathfrak{B}_1^2 \cap (\mathfrak{B}_2^2)^\perp = \dim \mathfrak{B}_2^2 \cap (\mathfrak{B}_1^2)^\perp.$$

Proof. We will show that under the assumption that condition 1 holds (equivalently, $(G_2^* G_1)^{-1} \in R\mathcal{L}_\infty$), condition 2 is equivalent with $\dim(\mathfrak{B}_1 \cap \mathfrak{B}_2^*)_{\text{stab}} = \mathfrak{n}(\mathfrak{B}_1)$. Indeed, it was shown in the previous subsection that

$$(10.1) \quad \dim(\mathfrak{B}_1 \cap \mathfrak{B}_2^*)_{\text{stab}} \cap (\mathfrak{B}_1 \cap \mathfrak{B}_2^*)^+ = \dim \mathcal{X}_-(A - BD^{-1}C) \cap \mathcal{X}_+(A)$$

and

$$\dim(\mathfrak{B}_2 \cap \mathfrak{B}_1^*)_{\text{stab}} \cap (\mathfrak{B}_2 \cap \mathfrak{B}_1^*)^+ = \dim \mathcal{X}_-(-(A - BD^{-1}C)^\top) \cap \mathcal{X}_+(-A^\top).$$

Now note that

$$\begin{aligned} & \mathcal{X}_-(-(A - BD^{-1}C)^\top) \cap \mathcal{X}_+(-A^\top) \\ &= \mathcal{X}_-(A - BD^{-1}C)^\perp \cap \mathcal{X}_+(A)^\perp = (\mathcal{X}_-(A - BD^{-1}C) + \mathcal{X}_+(A))^\perp. \end{aligned}$$

As a consequence we obtain

$$(10.2) \quad \dim(\mathfrak{B}_2 \cap \mathfrak{B}_1^*)_{\text{stab}} \cap (\mathfrak{B}_2 \cap \mathfrak{B}_1^*)^+ = n_1 + n_2 - \dim \mathcal{X}_-(A - BD^{-1}C) + \dim \mathcal{X}_+(A).$$

Next,

$$\begin{aligned} & n_1 + n_2 - \dim \mathcal{X}_-(A - BD^{-1}C) + \dim \mathcal{X}_+(A) \\ &= n_1 + n_2 - (\dim \mathcal{X}_-(A - BD^{-1}C) + \dim \mathcal{X}_+(A) \\ &\quad - \dim \mathcal{X}_-(A - BD^{-1}C) \cap \mathcal{X}_+(A)) \\ &= n_1 - \dim \mathcal{X}_-(A - BD^{-1}C) + \dim \mathcal{X}_-(A - BD^{-1}C) \cap \mathcal{X}_+(A). \end{aligned}$$

This is equal to $\dim \mathcal{X}_-(A - BD^{-1}C) \cap \mathcal{X}_+(A)$ if and only if the condition $n_1 = \dim \mathcal{X}_-(A - BD^{-1}C)$ holds; equivalently, $\dim(\mathfrak{B}_1 \cap \mathfrak{B}_2^*)_{\text{stab}} = \mathfrak{n}(\mathfrak{B}_1)$. \square

Thus, as indicated before in [21, p. 41], under the assumption that the distance in the \mathcal{L}_2 -metric is less than 1, the distance in the V-metric is less than 1 if and only if the dimensions of the intersections $(\mathfrak{B}_1 \cap \mathfrak{B}_2^*)_{\text{stab}} \cap (\mathfrak{B}_1 \cap \mathfrak{B}_2^*)^+$ and $(\mathfrak{B}_2 \cap \mathfrak{B}_1^*)_{\text{stab}} \cap (\mathfrak{B}_2 \cap \mathfrak{B}_1^*)^+$ are equal, whereas the distance in the Z-metric is less than 1 if and only if these intersections have dimension zero.

10.2. State space characterizations. In this final subsection we will collect the relevant material from section 9 and formulate state space conditions for the distance in our metrics to be less than 1, in terms of the minimal driving variable representations of \mathfrak{B}_1 and \mathfrak{B}_2 obtained by realization of G_1 and G_2 . We will first establish such conditions for the \mathcal{L}_2 -metric.

THEOREM 10.4. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$ with $\mathfrak{m}(\mathfrak{B}_1) = \mathfrak{m}(\mathfrak{B}_2)$. Then $d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) < 1$ if and only if the following two conditions hold:*

1. $D = D_2^\top D_1$ is nonsingular,
2. $A - BD^{-1}C$ has no imaginary axis eigenvalues.

Proof. (\Rightarrow) By Lemma 8.1, $G_2^* G_1$ is biproper. This implies $D = D_2^\top D_1$ is nonsingular. By Theorem 8.3, $\mathfrak{B}_1 \cap \mathfrak{B}_2^*$ is autonomous and its dimension is equal to $\mathfrak{n}(\mathfrak{B}_1) + \mathfrak{n}(\mathfrak{B}_2)$. Thus (9.13) is a minimal state representation, so it is observable. Finally, again by Theorem 8.3, $\mathfrak{B}_1 \cap \mathfrak{B}_2^*$ has no imaginary frequencies, so none of the eigenvalues of $A - BD^{-1}C$ can lie on the imaginary axis.

(\Leftarrow) $D = D_2^\top D_1$ nonsingular implies that $G_2^* G_1$ is biproper. By Lemma 8.2, $\mathfrak{B}_1 \cap \mathfrak{B}_2^*$ is autonomous and its dimension is equal to $\mathfrak{n}(\mathfrak{B}_1) + \mathfrak{n}(\mathfrak{B}_2)$. Also, (9.13) is a state representation of $\mathfrak{B}_1 \cap \mathfrak{B}_2^*$. Since $A - BD^{-1}C$ has no eigenvalues on the imaginary axis, $\mathfrak{B}_1 \cap \mathfrak{B}_2^*$ has no imaginary frequencies. Theorem 8.3 then yields $d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) < 1$. \square

Next, we turn to the V-metric again.

THEOREM 10.5. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$ with $\mathfrak{m}(\mathfrak{B}_1) = \mathfrak{m}(\mathfrak{B}_2)$. Denote $n_1 := \mathfrak{n}(\mathfrak{B}_1)$. Then $d_V(\mathfrak{B}_1, \mathfrak{B}_2) < 1$ if and only if the following conditions hold:*

1. $d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) < 1$,
2. $\dim \mathcal{X}_-(A - BD^{-1}C) = n_1$.

Proof. If $d_V(\mathfrak{B}_1, \mathfrak{B}_2) < 1$, then $d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) < 1$. Under this condition $\mathfrak{B}_1 \cap \mathfrak{B}_2^*$ is autonomous and by Lemma 9.7, $\dim(\mathfrak{B}_1 \cap \mathfrak{B}_2^*)_{\text{stab}} = \dim \mathcal{X}_-(A - BD^{-1}C)$. By Theorem 8.6 the latter equals n_1 . The converse is proved in a similar way. \square

Finally, we give a characterization for the Z-metric.

THEOREM 10.6. *Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}_{\text{cont}}^q$ and $\mathfrak{m}(\mathfrak{B}_1) = \mathfrak{m}(\mathfrak{B}_2)$. Then $d_Z(\mathfrak{B}_1, \mathfrak{B}_2) < 1$ if and only if the following conditions hold:*

1. $d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) < 1$,
2. $\mathcal{X}_-(A - BD^{-1}C) \oplus \mathcal{X}_+(A) = \mathbb{R}^{n_1+n_2}$.

Proof. $d_Z(\mathfrak{B}_1, \mathfrak{B}_2) < 1$ if and only if $d_{\mathcal{L}_2}(\mathfrak{B}_1, \mathfrak{B}_2) < 1$ and conditions 2 and 3 of Theorem 10.2 hold. By (10.1), condition 2 is equivalent with $\mathcal{X}_-(A - BD^{-1}C) \cap \mathcal{X}_+(A) = \{0\}$, and by (10.2) condition 3 is equivalent with $\mathcal{X}_-(A - BD^{-1}C) \cap \mathcal{X}_+(A) = \mathbb{R}^{n_1+n_2}$. \square

Note that since $\dim \mathcal{X}_+(A) = n_2$, the condition $\mathcal{X}_-(A - BD^{-1}C) \oplus \mathcal{X}_+(A) = \mathbb{R}^{n_1+n_2}$ implies that $\dim \mathcal{X}_-(A - BD^{-1}C) = n_1$. This indeed confirms that $d_Z(\mathfrak{B}_1, \mathfrak{B}_2) < 1$ implies $d_V(\mathfrak{B}_1, \mathfrak{B}_2) < 1$, as we already knew.

11. Conclusions. In this paper we have studied notions of distance between linear differential systems. We have introduced four metrics on the space of all controllable behaviors. Three of these have been defined in terms of gaps between closed subspaces of the Hilbert space $\mathcal{L}_2(\mathbb{R})$. After having established the relation between rational representations of behaviors and classical multiplication operators, we have expressed these metrics in terms of the proper rational matrices appearing in the rational representations. We have introduced a fourth metric on the space of controllable behaviors as a generalization of the ν -metric. As in the input-output framework, this definition has been given in terms of rational representations. For this metric, we have established a representation-free, behavioral characterization as well. We have

also made a comparison between the four metrics and have compared the values they take and the topologies they induce. Finally, for all metrics we have made a detailed study of necessary and sufficient conditions under which the distance between two behaviors is less than one. For this, both behavioral as well as state space conditions have been derived in terms of driving variable representations of the behaviors.

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